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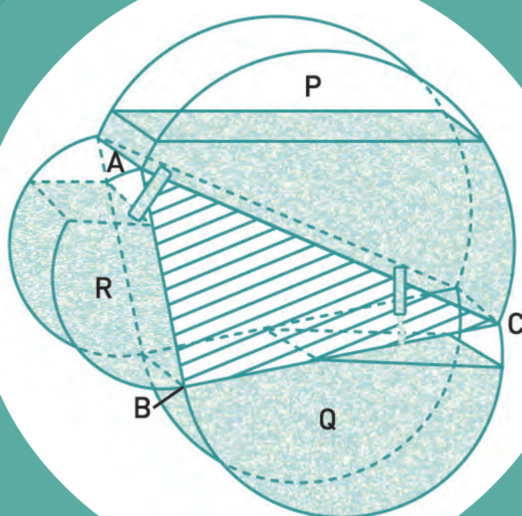

This figure represents generalization of Pythagoras Theorem

To Our Contributors

School Science is a journal published quarterly by the National Council of Educational Research and Training, New Delhi. It aims at bringing within easy reach of teachers and students the recent developments in science and mathametics and their teaching, and serves as a useful forum for the exchange of readers' views and experiences in science and mathametics education and science projects.

Articles suitable to the objectives mentioned above are invited for publication. An article sent for publication should normally not exceed ten typed pages and it should be exclusive to this journal. A hard copy of the article including illustrations, if any, along with a soft copy in CD should be submitted. Photographs (if not digital) should be at least of postcard size on glossy paper and should be properly packed to avoid damage in transit. The publisher will not take any responsibility or liability for copyright infringement. The contributors, therefore, should provide copyright permission, wherever applicable, and submit the same along with the article.

Manuscripts with illustrations, charts, graphs, etc. along with legends, neatly typed in double space on uniform sized paper, should be sent to Executive Editor, School Science, Department of Education in Science and Mathematics, NCERT, Sri Aurobindo Marg, New Delhi 110016.

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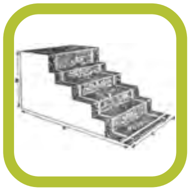

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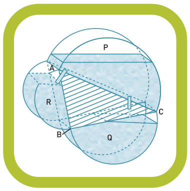

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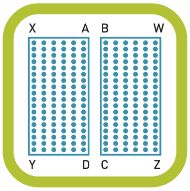

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EDITORIAL

Mathematics is a study of gravity (number), structure, space and change. The present issue is third in the series on the completion of 50 years of the Journal. We have included the articles from various disciplines of mathematics, which will help imparting knowledge, skills, values, etc. in mathematics among the young readers.

In the article, "Srinivasa Ramanujan..." the author discusses the contribution made in the field of mathematics by Ramanujan who was a genius with natural talent and had an unbelievable intuition for mathematical results. His great power of concentration ranked him as the greatest mathematician of all times.

The series of the three research articles– Magic Squares-I, Magic Squares-II and Magic Squares-III is an amusing read in which the author talks about the rules for development of magic squares in a very interesting way.

"Learning through Riddles" is an interesting article in which the author proves some of the mathematical riddles. And the article, "The Mysterious Infinity" discusses the concept of endlessness of the number, i.e., infinity.

The article, "Danger of Improper Use of Mathematical Results, Formulas and Symbols: Algebra" shows that mathematics is not a subject of mere memorisation and it needs a lot of thinking power and a very serious study, by listing the examples like law of indices, inequations, modulus sign, logarithm, binominal expansion, cancellation of a common factor from an equation, determinants, which may arise many fallacious results due to wrong use of formulas, results, etc. in algebra.

In the article, "Danger of Improper Use of Mathematical Results, Formulas and Symbols: Limit (Calculus)" the author discusses some points which may cause false results. The error may occur due to limit of the sum of functions, product of functions and negligence of symbols.

The article, "Danger of Improper Use of Mathematical Results, Formulas and Symbols: Integral (Calculus)" reviews some of the errors which may lead to false results in calculus. These errors may occur due to: the fundamental theorem of integral calculus; the neglect of the sign of an expression under a square root; making no distinction between proper and improper integrals; making the proper integral improper by some operation; improper substitution in definite integrals.

In the article, "Danger of Improper Use of Mathematical Results, Formulas And Symbols: Trigonometry", the author discusses the examples in trigonometry, equalisation of arguments in trigonometry, choosing the scale in drawing graphs of trigonometric functions, restrictions involved in inverse trigonometric functions, which may lead to many false results due to wrong use of formulae.

Pythagoras was perhaps the first to find a proof of the theorem considering the areas of the square on the sides of a right-angle triangle. In the article "Experimenting with

Pythagoras Theorem" the author discusses about the theorem and the expansion of it.

The aricle, "Exploring Mathematics through Origami" describes the general principle of mathematics that is involved in creation of shapes through folding of paper. These principles can be easily explained through folding of a square sheet of paper. Thus, difficult concepts can be explained to students using their imagination and creativity.

The review article, "The Principle and the Method of Finding Out the Cube Root of any Number" has also been added in which the researcher explains the method to find the cube root of any number by the method of division. The advantage of this method is that it enables to find out the cube root of any number up to the desired decimal places.

We sincerely hope that our readers would find the articles interesting and educative. Your valuable suggestions, observations and comments are always a source of inspiration which guide us to bring further improvement in the quality of the journal.

MAGIC SQUARES - I

A. Venkatacharlu

(The author is a retired assistant headmaster of the Christian College School, Madras. He was a popular teacher of mathematics, who encouraged his students by providing challenging mathematical activities. He has continued his interest in mathematics till today. In this article, he presents his work on Magic Squares. It is hoped that both students and teachers will find it interesting.)

Nomenclature

1. Magic square: A square divided into the equal number of small squares both horizontally (i.e. length-wise) and vertically (i.e. breadth-wise) and filled up with positive whole numbers, in such a way that when the numbers in the different rows—vertical, horizontal and diagonal—are added, the same sum is obtained. While filling the small squares, no number should be repeated.

4	9	2
3	5	7
8	1	6

2. A house: Each small square is called a house.

3. Horizontal and vertical rows of houses: Houses lying in the same horizontal row, running from the left to the right, form a horizontal row of houses. Houses lying in the same vertical row, running from top to bottom, form a vertical row of houses.

The horizontal rows of houses are numbered from top downwards and the vertical rows of houses are numbered from the left to the right.

The houses themselves are numbered from left to right starting from the first horizontal row of houses and in the same order down to the end of the last horizontal row.

Note: Any corner house may be called House 1 and the numbering may proceed in either direction from this house.

The different ways of numbering in a four-house squares are shown on the next page.

4. A series: An arithmetical series is in short called a Series (Later, what I prefer to call a compound series is introduced).

Part 1

The rules for filling up a magic square with an *odd* number of houses each way:

Selection of Series

(1) It may be the one consisting of natural numbers starting with any number, e.g. 5,6,7,8,....(It may be in descending or ascending order).

(2) It may be arithmetical series with any common difference (C.D.) taken in either ascending or descending order, e.g., 5,8,11,14.......or 20,18,16....

(3) If it is a 'n' house square it may be 'n' sets of 'n' numbers of 'n' different series, the C.D. being the same in ALL and the 1st numbers of these n series forming a different series with different C.D. For example, let us take a 5 house square. The numbers chosen may be 2,5,8,11,14; 7,10,13,16,19; 12,15,18,21,24; 17,20,23,26,29; 22,25,28,31,34.

Rules for Filling the Squares

Method 1. Let us take the above compound series.

1. Enter the first number of the chosen series in the central house of the lowest horizontal row of houses.
2. After an entry move down diagonally to the right. While moving down diagonally you may (a) go out of the square and be in the line of a vertical or horizontal row of houses; (b) go out of a corner; (c) go into empty house, and (d) go into a house which is filled up.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

1	5	9	13
2	6	10	14
3	7	11	15
4	8	12	16

16	12	8	4
15	11	7	3
14	10	6	2
13	9	5	1

4	3	2	1
8	7	6	5
12	11	10	9
16	15	14	13

13	9	5	1
14	10	6	2
15	11	7	3
16	12	8	4

16	15	14	13
12	11	10	9
8	7	6	5
4	3	2	1

4	8	12	16
3	7	11	15
2	6	10	14
1	5	9	13

13	14	15	16
9	10	11	12
5	6	7	8
1	2	3	4

Note: The C.D. in each of the five sets in 3, and the first numbers of these sets form a different series with C.D. of 5.

(This is what I call a compound series)

(While using these, they should be used in order).

If you go out of the square, and are in a line with a row of houses, enter the next number in the house at the far end square of this rows of houses.

It you go out of a corner or into a house which has been filled up, enter the next number in the house just above the house you last filled up.

12	23	34	5	16
19	15	26	22	8
11	7	18	29	25
28	14	10	21	17
20	31	2	13	24

If you go into a vacant house, enter the next number there.

Note:

(i) You may begin in the central house of any one of the other three last rows of the houses.

(ii) Go diagonally to the left.

If you do so, you must adopt the other rules accordingly.

(iii) Horizontal rows at equal distances from the central row may be interchanged.

Vertical rows at equal distances from the central row may be interchanged. Or both.

Method 2.

(1) Enter the first number of the series in the house just below the central house.

(2) Go down diagonally to the right. (Note: you may go to the left also).

(3) If you go into a vacant house enter the next number there.

(4) If you go out of the full square, enter the next number in the house at the other end of the row of houses.

(5) If you go into a house which has been filled up, enter the next number in the house two houses below the house you last filled up and enter the next number there and if this is not possible go (n-2) houses above the house you last filled up, and enter the next number there.

(6) If you get out of a corner, go (n-2) houses above the house you last filled up and enter the next number there.

You may start with the house just above the central house or to the right or left of the central house. But you must adopt the rules accordingly.

An Example: Let us take a 7-house square and the following compound series.

1,3,5,7,9,11,13; 8,10,12,14,16,18,20;

15,17,19,21,23,25,27; 22,24,26,28,30,32,34;

29,31,33,35, 37, 39,41; 36, 38, 40,42,44,46,48;

43,45,47,49,51,53,55.

22	51	17	46	12	41	7
9	24	53	19	48	14	29
31	11	26	55	21	36	16
18	33	13	28	43	23	38
40	20	35	1	30	45	25
27	42	8	37	3	32	47
49	15	44	10	39	5	34

This method is the same as shown below by a 7-house sq. with natural numbers.

Rules Followed

Write down the series chosen in order to form a seven-house square (no sq. is drawan). Draw lines diagonally both ways as shown. Find out the central house formed by the rows. Keeping this as the central house formed by the rows. Keeping this as the central house of the Magic Square (25 is in the central house) mark the Magic Square.

In a line with the central house with 25 you find 1 and 49 outside and two vacant houses inside. Count 7 houses (changing this according to the number of the houses in one line of the M.S.) along the row in which 1 is situated and enter it there. Similarly, enter all the other numbers found outside the square in the 7th house counted from the house in which it is placed.

A 9-house square with algebraic symbol

Initial number is 'a'

C.D. throughout is d.

C.D. between the numbers of the 9 sets=C

a+4c	a+5c+2d	a+6c+4d	a+7c+6d	a+8c+8d	a + d	a+c+3d	a+2c+5d	a+3c+7d
a+3c+8d	a+4c+d	a+5c+3d	a+6c+5d	a+7c+7d	a+8c	a+2d	a+c+4d	a+2c+6d
a+3c+7d	a+3c	a+4c+2d	a+5c+4d	a+6c+6d	a+7c+8d	a+8c+d	a +3d	a+c+5d
a+c+6d	a+2c+8d	a+3c+d	a+4c+3d	a+5c+5d	a+6c+7d	a+7c	a+8c+2d	a +4d
a+5d	a+c+7d	a+2c	a+3c+2d	a+4c+4d	a+5c+6d	a+6c+8d	a+7c+d	a+8c+3d
a+8c+4d	a+bd	a+c+8d	a+2c+d	a+3c+3d	a+4c+5d	6+5c+7d	a+6c	a+7c+2d
a+7c+3d	a+8c+5d	a+7d	a+c	a+2c+2d	a+3c+4d	a+4c+6d	a+5c+8d	a+6c+d
a+6c+2d	a+7c+4d	a+8c+6d	a+8d	a+ c +d	a+2c+3d	a+3c+5d	a+4c+7d	a+5c
a+5c+d	a+6c+3d	a+7c+5d	6+8c+7ds	a	a+c+2d	a+2c+4d	a+3c+6d	a+4c+8d

Sum = 9a + 3d + 36c

Method 1 is followed. Method 2 may also be tried.

Something Interesting

In the 1st Magic Square above, (1) the number in the central house 18 multiplied by 5, the number of houses each way gives 90, which is the sum of each line.

2. 12+24= 36. Now go round say, in the clockwise direction starting from 12+24 In the next stage are 23 and 13 which when added gives 36. Trace similar results in the next inner round. 36 is also 2×18, i.e., twice the number in central square.

   In the second filled-up magic square find out similar results.

MAGIC SQUARES - II

A. Venkatacharlu

To Fill A (4n +2) House Magic Square

The Ist one of this kind is a 6-house-sqaure when n is 1. The others are with houses numbering 10, 14, 18 and so on.

Let us start with a 6-house-sqaure. If natural numbers (a series can also be taken) are taken and are entered in order in the 36 houses of a 6-house-square, the numbers found in the outer most squares are 1 to 6, 12, 18, 24, 30, 36, 35, 34, 33, 32, 31, 25, 19, 13 and 7 and the numbers that remain inside squares are 8 to 11, 14 to 17, 20 to 23 and 26 to 29. Using these, fill up the inside 4-house-square of a 6-house-square. A 6-house-square filled up with natural number as described is shown below. It is not a magic square.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

The inner 4-house-square filled up by 8, 10, 11, etc., can be arranged to form magic square.

8	28	27	11
23	15	16	20
17	21	22	14
26	10	9	29

The square gives us a total of 74 and we have enough pairs of complements, e.g. 1, 36;2, 35; and so on, each pair giving a total of 37, to fill up the houses all round outside and give a total of 111 c.c.

$$\frac{37 \times 36}{2 \times 6} = 111$$

Let the corner pairs remain in their places.

The pairs in the first and last horizontal lines are*

1	2	3	4	5	6
36	35	34	33	32	30
31					36

*The corner numbers shown inside circles are left out

They cannot be used as they are, since in the lower line there are 6 numbers in thirties and none in the top line. Therefore, this is how it is done.

1	2	3	4	5	6
31	35	34	33	32	30

Rearrange the numbers as shown above by arrows. We have

1	2	34	33	5	6
31	35	3	4	32	36

Two thirties alone have been sent.

To send one more make the following change

2	34	33	5
35	3	4	32

We now have

2	34	33	35
32	3	4	5

2 and 32 are not complements

5 and 35 are not complements

2+32 = 34 and it is 3 less than 37

5+35 = 40 and it is 3 more than 37

∴ 2 and 32 and 35 and 5 should go into such vertical lines in which, in the horizontal lines which cuts these vertical lines are numbers differing by 3.

In the 3rd horizontal lines, we have 17 and 14 in the 2nd and 5th houses. Interchange them and enter 2 and 32 in the 5th vertical lines and 35 and 5 in the 2nd vertical line. The other two pairs can be entered into the two remaining vertical lines as we like.

Now for two vertical lines

7	13	19	25
30	24	18	12

Every number in the lower line is 5 more than one number in the top line. To set this right make a change similar to the above.

7	**13**	**19**	**25**
30	**24**	**18**	**12**
7	24	18	25
30	13	19	12

We have, thus, sent only two numbers. To send one more from the lower line which is 5 more than one in the top line rearrange them as above in the horizontal lines. (Connect 25 and 30 diagonally).

7	24	18	25
30	13	19	12

After the change we have

7	24	18	30
12	13	19	25

7 and 12 and 30 and 25 are not complements

7+12 = 19 and it is less than 37

30+25=55 and it is 18 more than 37

In the 3rd vertical lines we see 28 in the 2nd house and 10 in the 5th house. Interchange them and enter 7 and 12 in the 5th horizontal line and enter 30 and 25 in the 2nd horizontal line.

1	35	34	33	2	6
30	8	10	27	11	25
24	23	15	16	20	13
18	14	21	22	17	19
7	26	25	9	29	12
31	5	3	4	32	36

The 6-house square is now filled up.

10-house square

(The changes alone are give and no full explanation)

2 3 4 5 6 7 8 9

99 98 97 95 94 93 92

2 3 97 96 95 94 8 99

99 98 4 5 6 7 93 92

2 3 97 96 95 94 8 99

92 98 4 5 6 7 93 9

94 108

94 is 7 less than 101

108 is 7 more than 101

In the 2nd horizontal line, in the 5th and 6th houses are 15 and 22 differing by 7. Interchange them. Enter 2 and 92 in the 5th Vertical line and 9 and 99 in the 6th vertical line –

11 21 31 41 51 61 71 81

90 80 70 60 50 40 30 20

11 21 70 60 50 40 71 81

90 80 31 41 51 61 30 20

11 21 70 60 50 40 71 90

81 80 31 4 51 61 30 20

92 110

92 is 9 less than 101 and 110 is 9 more than 110. In the 8th vertical line 57 and 48 with a difference of 9. Interchange them and enter 11 and 81 in the 7th horizontal line and 90 and 20 in the 6th horizontal line.

1	3	97	96	2	99	95	94	8	10
21	12	88	87	28	22	78	77	25	80
70	85	17	18	82	75	27	28	72	31
60	19	83	84	16	29	73	74	26	41
50	86	14	13	89	76	24	23	79	51
90	32	68	67	35	42	58	57	45	20
11	65	37	38	62	55	47	48	52	81
40	39	63	64	36	49	53	54	46	61
71	66	34	33	69	56	44	43	59	30
91	98	4	5	92	9	6	7	93	100

Note the change between the method followed under a 6-house sq. and this. This is followed for 14, 18, 22..... house squares. This is easier to find out internal changes.

A 10-house square with Algebraic Symbols

Initial number is 'a' and C, D is 'd'

a	a + 2d	a + 96d	a + 95d	a + d	a + 98d	a + 94d	a + 93d	a + 7d	a + 9d
a + 20d	a + 11d	a + 87d	a + 86d	a + 14d	a + 21d	a + 77d	a + 76d	a + 24d	a + 79d
a + 69d	a + 84d	a + 16d	a + 17d	a + 81d	a + 74d	a + 26d	a + 27d	a + 71d	a + 30d
a + 59d	a + 18d	a + 82d	a + 83d	a + 15d	a + 28d	a + 72d	a + 73d	a + 23d	a + 40d
a + 49d	a + 85d	a + 13d	a + 12d	a + 88d	a + 75d	a + 23d	a + 22d	a + 78d	a + 50d
a + 89d	a + 31d	a + 67d	a + 66d	a + 34d	a + 41d	a + 57d	a + 56d	a + 44d	a + 19d
a + 10d	a + 64d	a + 36d	a + 37d	a + 61d	a + 54d	a + 46d	a + 47d	a + 51d	a + 80d
a + 39d	a + 38d	a + 62d	a + 63d	a + 35d	a + 48d	a + 52d	a + 53d	a + 45d	a + 60d
a + 70d	a + 65d	a + 33d	a + 32d	a + 68d	a + 55d	a + 43d	a + 42d	a + 58d	a + 29d
a + 90d	a + 97d	a + 3d	a + 4d	a + 91d	a + 8d	a + 5d	a + 6d	a + 92d	a + 99d

MAGIC SQUARES - III

A. Venkatacharlu

Filling a 4-house-square

We may choose any of the following series:

1. It may be any 16 consecutive natural numbers e.g., 5, 6,...............20.
2. It may be 16 consecutive numbers of an arithmetical series with a common difference e.g., 1, 3, 5,........31.
3. It may be 4 sets of 4 consecutive numbers of an arithmetical series separated by equal intervals e.g., 1, 3, 5, 7; 13, 15, 17, 19; 25, 27, 29, 31; 37, 39, 41, 43.
4. It may be 2 sets of 8 consecutive numbers of a series separated by an interval of any length e.g. 1,3,5,...15; 21,23,25............35.
5. It may be 4 sets of 4 consecutive numbers of 4 different series, the C.D. being the same in all and the difference between the first numbers of the 1st and 2nd sets being equal to the difference between the 1st numbers of the 3rd and 4th sets. The last of the 2nd and the 1st of the 3rd may be separated by any length. E.g.,1,3,5,7; 10,12,14,16; 50,52,54,56; 59,61,63,65.
6. It may be 4 sets of 4 consecutive numbers of 4 different series, the C.D. being the same in all and the 1st numbers of the 4 sets forming a different series with a different C.D. e.g., 1,3,5,7; 10,12,14,16; 19,21, 23, 25; 28,30,32,34.

We have already given numbers to the houses in a 4-house-square. The 1st method has been followed in the forthcoming 4-house-squares. While filling the square follow the order given by these numbers.

How to fill up a 4-house-square

Enter the 1st, the 4th, the 13th and 16th numbers in their respective corner houses; the 6th, 7th , 10th and 11th numbers in their respective central houses. Enter the 2nd and 3rd numbers in the 15th and 14th houses, the 5th and 9th numbers in the 12th and 8th houses, the 8th and 12th numbers in the 9th and 5th houses and the 14th and 15 numbers in the 3rd and 2nd houses. (With sufficient practice this becomes automatic).

4-house-squares filled up with the series given above:

(1) Series (i) given above, starting from the left hand top corner and proceeding to the right (2) Series 2 given above, starting from the right hand top corner and proceeding to the left (3) Series 3 given above. Starting from the right bottom corner and proceeding to the left.

5	19	18	8
16	10	11	13
12	14	15	9
17	7	6	20

(1)

7	27	29	1
17	13	11	23
9	21	19	15
31	3	5	25

(2)

43	3	5	39
13	29	27	19
25	17	15	31
7	39	41	1

(3)

7	21	9	35
31	13	25	3
33	11	23	5
1	27	15	29

(4)

28	5	3	34
16	21	23	10
24	12	14	19
1	32	30	7

(5)

(4) Series 4 and starting the left bottom corner and proceeding upwards (6) Series 6 and starting from the left bottom corner and proceeding to the right. Series 5 given in algebraic symbols: a, a+d, a+2d, a+3d; a + b, a + b+ d, a+b+2d, a+b+3d; p, p + d, p+2d, p+3d; and p + b, p + b + d, p+b+2d, p+b+3d is filled in the square below:

a	p+b+2d	p + b +d	a+3d
P+3d	a +b +d	a+b+2d	p
a+b+3d	p + d	p+2d	a + b
p + b	a+2d	a + d	p+b+3d

Sum = 2a+2b+2p+6d

Note: After filling a square

(1) 1st and 4th horizontal lines may be interchanged.
(2) 1st and 2nd and 3rd and 4th lines may be interchanged.
(3) 1st and 3rd and 2nd and 4th lines may be interchanged
(4) 2nd and 3rd and 2nd and 4th lines may be interchanged.

Similarly with the vertical lines, for change 1 in the horizontal lines 4 changes in the vertical lines can be made similarly for changes 2,3, and 4.

Similarly for change 1 in the vertical lines 4 changes in the horizontal lines can be made. These changes will give us a large number of arrangements inspite of a few overlappings.

Something Special

Suppose you want to engrave the year of construction of a building in the 2nd and 3rd houses taken together of a 4-house-square (as is seen in one building in Greece) and put it in a prominent place, here is the method.

Let us take the year 1976. We often read it as nineteen seventy-six. 19 and 76 are two far from each other to belong to one series. Therefore 19 must belong to one series of 8 numbers and 76 to another of 8 numbers, both the series being with the same C.D. Here are 4 ways of doing it with 1 as C.D.

1. 26, 25......19; 79, 78.......72
2. 12, 13......18, 19; 73,......80
3. 15, 16......22; 76,.......83
4. 23, 22......16; 76, 75........69

(1)

75	19	76	26
24	78	21	73
25	77	20	74
72	22	79	23

(Descending Order)

(2)

77	19	76	12
14	74	17	79
13	75	18	78
80	16	73	15

(Ascending Order)

(3)

83	19	76	18
16	78	21	81
17	77	20	82
80	22	79	15

(Ascending Order)

(4)

69	19	76	20
22	74	17	71
21	75	18	70
72	16	73	23

(Descending Order)

In the above the starting point and the direction are shown by arrows—whether the series is of ascending order or descending order can easily be found. Only natural numbers are used. With a C.D. of 2 only one series in the ascending order is possible. It is 5, 7, 9,11,15,17, 19; 70, 72, 74, 76, 78, 80, 82, 84. It must be started from right hand top corner and proceed downwards.

If a series is of descending order any C.D. can be used. Many series are possible. Only 3 are given below.

(1) 33, 31, 29, 27, 25, 23, 21, 19; 82, 80, 78, 76, 74, 72, 70, 68

C.D. is 2

(2) 40,37,.....19; 85,82,79,....64

C.D. is 3

(3) 47,43,39........19; 88,84,.....64,60

C.D. is 4

4th house is the starting point for these and we should proceed downwards.

Using the principle involved in series 5 given before we can have many other series. Only one is given here:

14,13,12,11; 22,21,20,19; 79,78,77,76; 87,86,85,84.

87	19	76	14
12	78	21	85
13	77	20	86
84	22	79	11

Something interesting

At the end of Part I, under a similar heading, we saw interesting things. Trace similar ones in all the above 4-house-squares.

Filling up a magic square of houses each way being a multiple of four.

Let us take a 12-house –square.

Pair the 1st and last i.e., 1 and 144, and second and the last but one i.e., 2 and 143 and so on. We will have 72 pairs.

Divide the 12-house-squares into nine 4-house-squares. Fill up any one of these 4-house-squares with the first 8 pairs taking the numbers in the ascending order e.g.,1,2,3,4,5,6,7,8,137,138,139,140,141,142,143,144. They can be taken in the descending order also.

1	143	142	4
140	6	7	137
8	138	139	5
141	3	2	144

Since each square gives the same total one square may be filled up in the ascending order and another in the descending order as one likes.

1	143	142	4	9	135	134	12	17	127	126	20
140	6	7	137	132	14	15	129	124	22	23	121
8	138	139	5	16	130	131	13	24	122	123	21
141	3	2	144	133	11	10	136	125	19	18	128
25	119	118	28	33	111	110	36	41	103	102	44
116	30	31	113	108	38	39	105	100	46	47	97
32	114	115	29	40	106	107	37	48	98	99	45
117	27	26	120	109	35	34	112	101	43	42	104
49	95	94	52	57	87	86	60	65	79	78	68
92	54	55	89	84	62	63	81	76	70	71	73
56	90	91	53	64	82	83	61	72	74	75	69
93	51	50	96	85	59	58	88	77	67	66	80

Sum = 870

Sum in each 4-house-square = 290

All are filled up in ascending order only.

DANGER OF IMPROPER USE OF MATHEMATICAL RESULTS, FORMULAS AND SYMBOLS: ALGEBRA

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Introduction

Mathematics Education has two aspects: the first deals with how to teach and the second with what to teach. The first aspect is concerned with the methodology of teaching mathematics, the "how and why" of learning process, the discussion of the objectives of learning mathematics, preparation of question bank, teachers' guides and so on. The second aspect deals mainly with the content, i.e., the subject matter of mathematics. In actual practice, teachers are often found to be giving more attention to the first aspect. It appears that the second aspect of mathematics education has not been paid so much attention resulting in students' getting wrong notions and concept in the content area. Since in teaching mathematics to the students, only manipulatory aspect has generally been emphasised, mathematical rules, formulas, theorems, etc., are mechanically memorised by the students without giving importance to the aspects – the language, the thought process and the logic of the subject. As a result, students sometimes get used to mathematical results, formulas, etc. without giving a thought to their validity and restrictions on them, if any, thus arriving at some fallacious results and mistakes for which they hardly find proper explanation.

In a series of articles of which this is the first, these types of improper use of results, formulas, etc., from various branches of mathematics and their consequences will be discussed with the help of examples.

Laws of indices

From the laws of indices, students know that if $a^x = a^y$, then $x = y$.

Also if $a^x = b^x$, then $a = b$.

Now, a student of even plus two stage may perhaps become puzzled and cannot explain the reason when he is asked the following questions:

Since $a^x = a^y \quad x = y$,

Hence $1^1 = 1^2 = 1^3 = 1^4$ ....

$1 = 2 = 3 = 4 =$ ....

i.e., all numbers are equal – a result which is absurd. What is the reason of getting this absurd result?

Again, since $a^x = b^x$

$\Rightarrow$ a = b,

Hence $1^0 = 2^0 = 3^0 = 4^0 = \ldots\ldots.. = 1$

$\Rightarrow$ 1 = 2 = 3 = 4 = ...,

which is again an absurd result.

Why is this absurd result obtained?

Teachers generally don't mention or emphasise the restrictions involved in the above formulas of indices. In fact, these formulas are not true for all values of x, y or a, b. The formula $a^x = a^y \Rightarrow x = y$, for example, is valid when $a \neq 0$ and $a \neq 1$ and the formula $a^x = b^x \Rightarrow a = b$ is true only when $x \neq 0$. Most of the school students are not aware of these facts. We thus see that we get these absurd results due to the fact that we are not considering the limitations of the algebraic formulas. This is an example to show how the wrong use of a formula may lead one to get an absurd result.

Inequations

In solving an inequation of the type $x^2 \geq 1$, most of the students follow a wrong method to get the solution as $x \geq \pm 1$, which is wrong. Teacher generally don't emphasise this type of mistake. That $x \geq -1$ is wrong is obvious, since x = 1/2 (which is > -1) does not satisfy the inequation $x^2 \geq 1$. The correct solution will be $x \geq 1$ or $x \leq -1$, which can be obtained in the usual way of solving a quadratic inequation, i.e., by bringing 1 on the left side and factorising the expression on the left. Similarly, the solution of $x^2 \leq 1$ can be obtained in the above manner and not by merely taking the square root on both sides.

Sometimes students have the tendency to write ad > cb or bc > ad from the inequation a/b > c/d, without considering the signs of a, b, c and d. This is so because students are not generally warned by the teachers against this type of mistakes. In fact, we can write ad > cb, if b and d are of the same sign whatever be the signs of a and c. If, however, b and d are of opposite signs, then bc > ad is true. This is because we can multiply (hence divide) both sides of an inequation by a positive number without changing the signs of inequality and without affecting its solution.

The words "inequation" and "inequality" are sometimes wrongly treated as synonymous. The difference between these two is analogous to that between an equation and an identity. An inequation is satisfied by a particular set of values of the variables involved, whereas an inequality is true for all values of the variables. Thus $x + y \leq 1$ is an inequation, but $x^2 + y^2 \geq 2xy$ is an inequality. We "solve" an inequation whereas we "prove" an inequality. Thus, the usual question like "solve the inequality" is insignificant in the sense that every inequality involving a number of variables may have any arbitrary solution.

Modulus sign

Teachers sometimes give a wrong definition of modulus (absolute value) of a number. They generally explain the idea in the following way:

The modulus of a number means its positive value. Hence to get the modulus of a negative number, one should just remove the negative sign before it. The modulus of a number without any negative sign before it, is the number itself. Thus $|-2| = 2$, $|-x| = x$, $|x| = x$ etc. But the above concept is wrong in the sense that the rule is not applicable in the case of an algebraic number expressed by a variable. For example, if we assume that $|-x| = x$, then putting $x = -2$, we get $|2| = -2$, which is absurd. Again, if $|x| = x$, then also $|-2| = -2$ when $x = -2$. But this is also absurd. Thus, the correct and the most general definition of the modulus of any number, say x, is as follows:

$|x| = x$ if $x \quad 0$

$= -x$ if $x < 0$.

Logarithm

We sometimes use the formulas of logarithm without paying much attention to the restrictions involved in these formulas. As a result, we arrive at some absurd results due to this improper use of the formulas of logarithm.

For example, when we use the formula

$\log_b m^n = n \log_b m$

in some computation, we do not generally pay any attention about the signs of m, n and b.

Now, consider the following example:

Using the above formula, we have

$\log_{10} 16 = \log_{10} (-4)^2 = 2 \log_{10} (-4)$

Also, $\log_{10} 16 = \log_{10} 4^2 = 2 \log_{10} 4$

$\therefore \log_{10} (-4) = \log_{10} 4$

$\Rightarrow -4 = 4$

$\Rightarrow -1 = 1$

$\Rightarrow -2 = 2, -3 = 3, -4 = 4$, etc.

Thus, all positive numbers are equal to all negative numbers, which is obviously absurd. This absurd result is due to the improper use of the formula $\log_b m^n = n \log_b m$. It should be noted carefully that we can use this formula only when m (hence m^n) and b are positive and n is any real number.

Similarly, in the definition of $\log_n m$, we restrict m and n to be positive. If we are not careful about these restrictions, we may again come across with absurd results. For example,

$\log_{-1} 1 = 2$, since $(-1)^2 = 1$.

Also, $\log_{-1} 1 = 0$, since $(-1)^0 = 1$.

$\therefore 2 = 0$, which is absurd.

Consider another example:

We have $\log_{-1}(-1) = 3$, since $(-1)^3 = -1$.

Also, $\log_{-1}(-1) = 5$, since $(-1)^5 = -1$.

Hence $3 = 5$, which is again absurd. All these absurd results are due to the wrong use of the definition of $\log_b m$. In fact, the definition is based on the fact that both m and n must be positive.

In this connection, it can be stated that some students treat the numbers like $\overline{1}.235$ and -1.235 as synonymous. In simplifying an arithmetic expression with the help of a log table, a student may get an equation like $\log x = -1.782$ where x is the simplified value of the expression. He then sees antilog table for .782 and adjusts the decimal point to get the answer. But this procedure is completely wrong. The fact is that -1.782 and $\overline{1}.782$ are different. $\overline{1}.782$ means -1 + .782 i.e.,- .218 which is obviously different from -1.782.

The correct procedure to find x is, therefore, to make the mantissa part positive. The characteristic part may be positive or negative.

Thus log x = –1.782 = -2 + (2 – 1.782)

= –2 + .218 = $\bar{2}$.218

Hence, to get x, we should see antilog for .218 and adjust the decimal point. This type of common mistakes by many students should be brought to their notice by the teachers.

Binomial Expansion

In the binomial expansion of $(1+x)^n$ where n is a negative integer or a fraction, we very often use the formula of the expansion without paying heed to the restrictions on x. We thus write

$$(1+x)^{-1} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots \text{ to } \infty$$

$$(1-x)^{-1} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \text{ to } \infty$$

Students memorise these formulas without thinking whether these formulas are true for all values of x or not. Now, observe the danger of the wrong use of the above-mentioned formulas

Putting x = 2 in the formula

$$(1-x)^{-1} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \text{ to } \infty,$$

we get

$$(1-2)^{-1} = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \ldots \text{ to } \infty$$

$$\text{or, } -1 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \ldots \text{ to } \infty$$

Left hand side is a negative number and the right hand side is a positive number and these two are equal, which is absurd. This absurd result is due to the fact that the above expansion formula is not valid for $|x| \geq 1$. In fact, when n is a negative integer or a fraction, the binomial expansion of $(1+x)^n$ is valid only when $|x| < 1$. This fact should be emphasized by the teachers in their classes.

Cancellation of a Common Factor from an Equation

In solving an algebraic equation, the natural tendency of most of the students is to cancel the common factor from both side without giving a thought whether the value of the factor may be zero or not. This type of wrong practice is very common among many students, because teachers generally do not emphasise the danger of dividing a number by zero. Cancellation of a common factor from both sides of an equation means dividing both sides by that factor. Now, if this factor is zero, then we are not allowed to divide by it, for division by zero is meaningless in mathematics.

Consider the following problem:

Solve completely the following equation:

$x^3 + x^2 - 4x = 0$

Students generally solve the equation as follows:

Dividing both sides of the equation by x, we get

$x^2 + x - 4 = 0$

$$\Rightarrow x = \frac{-1 \pm \sqrt{1+16}}{2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{17}$$

But the above solutions are not the only solutions. x = 0 is also a solution of the given equation. Students will naturally miss this solution due to the fact that they have already divided both sides of the equation by x; in other words, they have unconsciously assumed the fact that $x \neq 0$ – an assumption which is not justified.

The correct method of solving the above equation is as follows:

From the given equation, we have

$x(x^2 + x - 4) = 0$

$\therefore$ Either $x = 0$ or, $x^2 + x - 4 = 0$.
The solution of $x^2 + x - 4 = 0$ has already been shown above.

Curious students may ask a natural question. If zero is treated to be a number, why is a student not allowed to divide a number by zero? To answer this question, teachers can give many examples to show that division by zero leads one to get absurd results in mathematics. One of many examples is as follows:

We know that the statement

$x^2 - x^2 = x^2 - x^2$

is always true.

The above statement implies

$x(x - x) = (x + x)(x - x)$

Dividing both sides by $x - x$, we get

$x = x + x$

or, $x = 2x$ or, $1 = 2$ (!)

This absurd result is due to division by $x - x$ which is zero as well as by x which may also be zero.

Extraneous Solution

In solving algebraic equations, students are satisfied if they get some solution of the equation by applying the usual methods. They hardly verify whether the solutions obtained satisfy the given equation/equations or not. As a result, they may sometimes get a wrong solution.

Consider, for example, the equation:

$\sqrt{2x - 1} + \sqrt{x} = 2$ .... (1)

where positive values of the square roots should always be taken.

Squaring both sides of the equation (1), we get

$3x - 1 + 2\sqrt{2x^2 - x} = 4$

$\Rightarrow 2\sqrt{2x^2 - x} = 5 - 3x$

Squaring again, we get

$4(2x^2 - x) = 25 + 9x^2 - 30x$

$\Rightarrow x^2 - 26x + 25 = 0$
$\Rightarrow (x - 25)(x - 1) = 0$
$\Rightarrow x = 25$ or 1.
Hence, the solutions obtained should be $x = 25$ and $x = 1$. But if we put $x = 25$ in (1), we see that it does not satisfy this equation. Hence, $x = 25$ cannot be the solution and $x = 1$ is the only solution. The solution $x = 25$ which is called the extraneous solution should always be discarded. Teachers should make it clear to the students that this type of extraneous solution may be found in solving equation involving square roots and hence students should always verify the solutions by substituting these in the original equation.

Consistency and Inconsistency of Dependent Equations

Students know that a system of equations having no solution is inconsistent and if one equation is obtained from the other by multiplying by a constant, then the two equations are dependent. From this, they may conclude that two dependent equations are inconsistent, which is wrong. Teachers should emphasise the fact that two dependent equations in fact represent the same equation. These equations will have either one (in case of one variable) or infinite number (in case of more than one variable) or solution. Hence, two dependent equations are always consistent. On the other hand, two inconsistent (linear) equations involving two variables x and y represent two parallel lines which cannot meet together at a finite distance from the origin.

Determinants

In finding the value of a determinant of any order, students sometimes make wrong use of the operations with the elements of rows or columns of the determinant.

Students are familiar with the following theorems:

Theorem 1: The value of a determinant is unaltered if to each element of one column (or row) is added a constant multiple of the corresponding element of another column (or row). The following is the extension of the above theorem:
We can add multiples of any one column (or row) to every other column (or row) and leave the value of the determinant unaltered.

The practice of adding multiples of columns or rows at random is liable to lead to error unless each step is checked for validity by appeal to the following theorem:

Theorem 2: The determinant of order n

is equal to the sum of the 2^n determinants

$$\begin{vmatrix} a_1 + A_1 & b_1 + B_1 & \cdots\cdots & k_1 + K_1 \\ a_2 + A_2 & b_2 + B_2 & \cdots\cdots & k_2 + K_2 \\ \vdots & \vdots & & \vdots \\ a_n + A_n & b_n + B_n & \cdots\cdots & k_n + K_n \end{vmatrix}$$

corresponding to the 2^n different ways of choosing one letter from each column.

Now consider a third order determinant:

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

One of the curious errors into which one is led by adding multiples of rows at random is a fallacious proof that $\Delta = 0$ whatever a_i b_i c_i $(i = 1, 2, 3)$ may be. For example, in the above determinant, subtract the second column from the first, add the second and third columns and add the first and the third columns. We thus get

$$\Delta = \begin{vmatrix} a_1 - b_1 & b_1 + c_1 & c_1 + a_1 \\ a_2 - b_2 & b_2 + c_2 & c_2 + a_2 \\ a_3 - b_3 & b_3 + c_3 & c_3 + a_3 \end{vmatrix}$$

which is always zero. For,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} - \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix},$$

all other determinants being zero.

Hence,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Thus we have shown that whatever be the elements of the determinant, the value of the determinant is zero — a result which is absurd. We can also proceed in the following way to show this absurd result:

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 - a_2 & b_1 - b_2 & c_1 - c_2 \\ a_2 - a_1 & b_2 - b_1 & c_2 - c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(Subtracting the elements in the second row from those in and then subtracting the elements is the first row from those in the second)

$$= - \begin{vmatrix} a_1 - a_2 & b_1 - b_2 & c_1 - c_2 \\ a_1 - a_2 & b_1 - b_2 & c_1 - c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

= 0, the two rows being identical.

All these fallacious results are due to the fact that we have wrongly used the theorems stated above. In fact, when we add any multiple of a column (or row) to other column (or row), the former column (or row) should be left as it is. If, on the other hand, the former column (or row) is again added with a constant multiple of the latter column (or row) at the same time, then we shall get such fallacious results. The correct way of using the theorems stated above is as follows:

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 - a_2 & b_1 - b_2 & c_1 - c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

We can now add with the elements of the second row any constant multiple of the first row or second row and write the sum in the second row and leave the first or second row as it is.

Thus,

$$\Delta = \begin{vmatrix} a_1 - a_2 & b_1 - b_2 & c_1 - c_2 \\ a_2 - a_3 & b_2 - b_3 & c_2 - c_3 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

We can further write

$$\Delta = \begin{vmatrix} a_1 - a_2 & b_1 - b_2 & c_1 - c_2 \\ a_2 - a_3 & b_2 - b_3 & c_2 - c_3 \\ a_3 + a_2 & b_3 + b_2 & c_3 + c_2 \end{vmatrix}$$

This determinant cannot always be zero.

Improper Use of Symbols

It has been observed that in addition to improper use of mathematical results and formulas, students very often do mistakes in using some mathematical symbols and teachers also do not bother about these mistakes. As a result, students' concept about these symbols remain vague even when they study advanced mathematics. For example, they don't realise the difference between the symbols = and ⇒. While they simplify a mathematical expression, they use ⇒ in place of = and when they solve some equations or inequations, they use = in place of ⇒. It should be pointed out clearly to the students that we can use the symbol ⇒ between two statements and the symbol = between two expressions. An equation or an inequation or an identity (which has two sides) is an example of a statement.

Conclusion

The list of example which has been mentioned above is only a few out of many fallacious results which may arise due to the wrong use of formulas, results, etc., in Algebra. The main objective of mentioning these fallacious results is to show that mathematics is not a subject of mere memorisation, it needs a lot of thinking power and a very serious study. To have superficial knowledge in a particular topic in mathematics is more harmful than to have no knowledge in it. Our present day mathematics syllabus is so heavy and haphazard that it is very difficult even for teachers to have thorough concept in all topics in the present day school mathematics. We can, therefore, realise how dangerous it is for the students to have wrong concept in mathematics from the teachers.

With a view to remove some of the wrong concepts in different topics in mathematics which are now being taught in secondary and higher secondary classes from the minds of the teachers, a series of such papers each for a particular topic in mathematics will appear in the subsequent issues of this Journal. The author will be highly pleased if at least a few teachers are benefitted from the perusal of these papers.

DANGER OF IMPROPER USE OF MATHEMATICAL RESULTS, FORMULAS AND SYMBOLS : TRIGONOMETRY

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The mistakes appearing in mathematics textbooks may often lead to defective teaching. Some of these common mistakes are conceptual mistakes in solving problems, in the statement of mathematical formulas or in steps involved in derivations. Often, the teachers fall victim to these errors and incorrectly solve a problem in the class, say by improper use of a formula, theorem, etc. This, in turn, perpetuates wrong ideas among young children. The author has selected a few of such mistakes and errors in trigonometry at the higher secondary stage. These examples are based on first-hand experience of the author with textbooks and classroom teaching practices for many years. In this article he has cited many interesting examples to explain some of these errors and mistakes in the teaching of trigonometry.

Trigonometry plays a very important part in mathematics. Students and teachers solve problems of trigonometry perhaps without much difficulty by using a number of trigonometric formulas and results. When they get the correct answer, they are fully satisfied, but when they get some incorrect answer or two or more answers for the same problem by using different techniques, they fail to give proper reasonings. They hardly think about the validity of these formulas and results in solving problems as a result of which they sometimes arrive at a conclusion which is physically impossible. In the actual classroom situation, students and teachers may sometimes come across this type of fallacious results for which they may find it difficult to get proper explanation. It should be noted that any fallacious result in any branch of mathematics may be due to the violation of some rules of mathematics. Some theorems, formulas or results in mathematics may be true under certain restrictions or conditions. In practical situation, we memorise these formulas, theorems, etc., giving the least importance of these restrictions and conditions.

Below an attempt is made to show with some examples that there is a danger of violating these restrictions and conditions in trigonometric formulas and results.

Equalisation of Arguments

Students have the tendency to equalise the arguments from a trigonometric equation. Thus if tan x = tan y, they conclude that x = y. That it is not always true will be clear from the following example:

$\tan x = \tan(\pi + x)$

$\therefore x = \pi + x \therefore \pi = 0$, a result which is absurd. The explanation is that all trigonometric functions are periodic functions. In the case of periodic functions, equalisation of arguments is not always permissible. In fact, equalisation of arguments from a trigonometric equation is allowed only when both the angles lie in the interval $(0, \pi/2)$.

Thus if $0 \leq A, B \leq \pi/2$,

then $\sin A = \sin B \Rightarrow A = B$,

$\tan A = \tan B \Rightarrow A = B$,

and so on.

Choosing the Scale in Drawing Graphs of Trigonometric Functions

In elementary mathematics, arguments in all trigonometric ratios are expressed either in radians or in degrees. In $y = \sin x$, for example, the unit of x is in radian. But sin x, i.e., y is a number.

In drawing the graph of say, $y = \sin x$, we usually choose the same scale on x-axis as in y-axis. The question is: Is it necessary to choose the same scale on both x and y axes? The answer is 'no' if x in sin x has a unit. This is because we represent numbers on y-axis and radians on x-axis. If one unit on a graph paper along y-axis represents 1, there is no reason to believe that the same 1 unit along x-axis will represent 1 radian. We are free to choose any number of units of the graph paper along the x-axis as 1 radian. In physics and chemistry, we come across with a lot of equations connecting two variables with different units, and in drawing the graphs we choose different scales along x and y-axes. For example, in Boyle's law equation: PV = constant, P has the unit of pressure and V has the unit of volume and hence in drawing the graph, we are free to choose different scales for P and V.

Restrictions Involved in Inverse Trigonometric Functions

It is known that every inverse trigonometric function is single-valued and hence is defined within its principal value branch. We use well-known formulas of inverse trigonometric functions without paying much attention to the possible restrictions for which these are valid. As a result, we may arrive at some fallacious results for which we may not find any explanation. These are discussed below with series of examples.

Example 1

Students are familiar with the well-known formula:

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}.$$

If we now put x = 1 and y = 2, then L.H.S. = $\tan^{-1}1 + \tan^{-1}2$ which is greater than 0. Also R.H.S. = $\tan^{-1}\left(\frac{3}{-1}\right) = \tan^{-1}(-3) = -\tan^{-1}3$ which is less than 0.

It follows that a negative non-zero number is equal to a positive non-zero number which is absurd. Why is it so if the above formula is correct? This is because the above formula is true under certain values of x and y such that $xy < 1$. If we neglect this restriction, we are likely to get the absurd result as shown above.

Example 2

The formula:

$2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}$ is also well-known.

When we use this formula in solving some problems, we do not generally bother about whether it is true for all x or not. As a result, we again come across some absurd results. It can be shown that the above formula is true only for $-1 \le x \le 1$. For, if we take, say $x = \sqrt{3}$, (>1), then

L.H.S. = $2\tan^{-1}$“3 = $2\pi/3$, whereas

$$\text{R.H.S.} = \sin^{-1}\left(\frac{2\sqrt{3}}{1+3}\right) = \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$$

Hence, L.H.S. ≠ R.H.S. for $x = \sqrt{3}$.

Example 3

We now consider another well-known formula, viz.,

$$2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}$$

This formula is also used without paying heed to the possible restrictions on x. That the result is not true for $x < 0$, can be shown by taking, say, $x = -\sqrt{3}$.

Then L.H.S. = $2\tan^{-1}(-\sqrt{3}) = -2\pi/3$

$$\text{R.H.S.} = \cos^{-1}\frac{1-3}{1+3} = \cos^{-1}\left(-\frac{1}{2}\right)$$

$= \pi - \cos^{-1}1/2 = \pi - \pi/3 = 2\pi/3$

Hence, L.H.S. ≠ R.H.S. for $x = -\sqrt{3}$

In fact, the given result is true for $0 \le x < \infty$

Example 4

The result $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$ is another important result which is true for $-1 < x < 1$. For $x = \pm 1$, the L.H.S. is defined whereas the R.H.S. is undefined. That the result is not true beyond the interval $-1 < x < 1$ can be shown by putting $x = \sqrt{3}$ on both sides.

Example 5

We very often use the following formula without imposing any restriction on x :

$\cot^{-1}x = \tan^{-1}1/x$, $\sec^{-1}x = \cos^{-1}1/x$, $\text{cosec}^{-1}x = \sin^{-1}1/x$.

In fact, the above results are true for $x \neq 0$. For $x = 0$, R.H.S. for each of the above equations is undefined whereas the L.H.S. is defined for $x = 0$.

Example 6

If we do not consider these restrictions on x while memorising the important formulas as mentioned above, we are likely to get some unexpected results. As an example,

$\sin^{-1}\left(2x\sqrt{1-x^2}\right)$ can be shown to be equal to $2\sin^{-1}x$ as well as $2\cos^{-1}x$ which are not identical functions.

We first put $x = \sin\theta$.

Then $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \sin^{-1}(2\sin\theta\cos\theta)$

$= \sin^{-1}\sin 2\theta = 2\theta = 2\sin^{-1}x$, ...(1)

Again, we put $x = \cos\theta$.

Then $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \sin^{-1}(2\sin\theta\cos\theta) = \sin^{-1}\sin 2\theta = 2\theta = 2\cos^{-1}x$. ....(2)

Why are we getting two different answers?

The explanation is that the result (1) is true when

$-1/\sqrt{2} \le x \le 1/\sqrt{2}$, whereas the result (2) is true when $0 \le x \le 1$.

Hence, both the results are true, but for different values of x. To explain how these two different sets of values of x are obtained in the two cases, we recall that the principal value branches of $\sin^{-1} x$ and $\cos^{-1} x$ are respectively $-\pi/2 \le \sin^{-1} x \le \pi/2$ and $0 \le \cos^{-1} x \le \pi/2$.

Now from (1), x should be such that

$$-\pi/2 \le \sin^{-1}\left(2x\sqrt{1-x^2}\right) \le \pi/2$$

or, $-\pi/2 \le 2\sin^{-1} x \le \pi/2$

or, $-\pi/4 \le \sin^{-1} x \le \pi/4$

or, $-1/\sqrt{2} \le x \le 1/\sqrt{2}$.

In the case of the equation (2), we note that the principal value branch of $\cos^{-1} x \ge 0$, but the principal value branch of $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$ may be negative also. Hence, taking the common region for which both the functions are defined, we have

$0 \le 2\cos^{-1} x \le \pi$

or, $0 \le \cos^{-1} x \le \pi/2$

or, $0 \le x \le 1$.

Example 7

Another example of getting a wrong answer to a problem due to the violation of the possible restriction on x while using a formula as mentioned above can be given.

Suppose, we are asked to solve the following equation:

$$\tan^{-1}\frac{2x}{1-x^2} + \cot^{-1}\frac{1-x^2}{2x} = \frac{\pi}{3}$$

The solution of the above equation can be obtained in the following manner:

From the given equation, we have

$$\tan^{-1}\frac{2x}{1-x^2} + \tan^{-1}\frac{2x}{1-x^2} = \frac{\pi}{3}$$

or, $2\tan^{-1}\frac{2x}{1-x^2} = \frac{\pi}{3}$

or, $\tan^{-1}\frac{2x}{1-x^2} = \frac{\pi}{6}$ (A)

If we now use the formula as stated in example 4 above, we get

$2\tan^{-1} x = \frac{\pi}{6}$

or, $\tan^{-1} x = \frac{\pi}{12}$

or, $x = \tan\frac{\pi}{12} = 2 - \sqrt{3}$

If we, however, do not use the above formula, then from (A), we get

$\frac{2x}{1-x^2} = \tan\frac{\pi}{6} = 1/\sqrt{3}$

or, $x^2 + 2\sqrt{3}x - 1 = 0$

or, $x = \frac{-2\sqrt{3} \pm \sqrt{12+4}}{2} = -\sqrt{3} \pm 2$

i.e., the values of x are $-\sqrt{3} + 2$ and $-\sqrt{3} - 2$. We, thus, see that due to the application of a formula, we lose some solution. Both the solutions obtained by avoiding the formula satisfy the original equation and hence these are the actual complete solution of the given equation.

The incomplete solution obtained as a result of the application of formula

$$2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$$

is due to the fact that the above formula is not true for all values of x. It is true only when $-1 < x < 1$ and that is why we did not get the solution $-\sqrt{3}-2$ which is obviously less than -1. This example shows that if we use a formula at random without thinking about the possible restrictions, we are likely to obtain incomplete and sometimes wrong answers.

Sometimes, due to the wrong use of a formula or due to the violation of possible values of x for which the formula holds, we get contradictory results. We consider an example:

Example 8

Simplify the expression:

$$\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$$

We solve the problem in several ways:

First Method

Writing $\sqrt{1+\sin x} = \sin\frac{x}{2}+\cos\frac{x}{2}$

and $\sqrt{1-\sin x} = \sin\frac{x}{2}-\cos\frac{x}{2}$, we get the given expression:

$$=\cot^{-1}\left(\frac{\sin\frac{x}{2}+\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}}{\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}+\cos\frac{x}{2}}\right)$$

$$=\cot^{-1}\tan\frac{x}{2} = \cot^{-1}\cot\left(\frac{\pi}{2}-\frac{x}{2}\right) = \frac{\pi}{2}-\frac{x}{2} \qquad \text{..... (B)}$$

Second Method

If we write

$$\sqrt{1+\sin x} = \cos\frac{x}{2}+\sin\frac{x}{2}$$

and $\sqrt{1-\sin x} = \cos\frac{x}{2}-\sin\frac{x}{2}$, then the given expression

$$=\cot^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}+\sin\frac{x}{2}}\right)$$

$$=\cot^{-1}\left(\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\right) = \cot^{-1}\cot\frac{x}{2} = \frac{x}{2} \qquad \text{.............. (C)}$$

Third Method

If we rationalise the denominator, then the given expression

$$=\cot^{-1}\left(\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{1+\sin x-1+\sin x}\right)$$

$$=\cot^{-1}\left(\frac{1+\sin x+1-\sin x+2\sqrt{1-\sin^2 x}}{2\sin x}\right)$$

$$=\cot^{-1}\left(\frac{1+\cos x}{\sin x}\right)$$

$$=\cot^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$

$= \cot^{-1}\cot\frac{x}{2}$

$= \frac{x}{2}$ .... (D)

We observe that the results (C) and (D) are the same but the result (B) is different. Why is it so? It is because in all the three methods of solution, the trigonometric formulas have been randomly used without giving a thought to the possible restrictions on x.

In the first two methods, for example, the results:

$\sqrt{1 \pm \sin x} = \sin\frac{x}{2} \pm \cos\frac{x}{2}$

and $\sqrt{1 \pm \sin x} = \cos\frac{x}{2} \pm \sin\frac{x}{2}$

have been used without considering the signs of $\sin\frac{x}{2} \pm \cos\frac{x}{2}$ and $\cos\frac{x}{2} \pm \sin\frac{x}{2}$.

We know that:

$\sqrt{x^2} = x$, if $x \geq 0$

$= -x$, if $x \leq 0$.

Hence, $\sqrt{1+\sin x} = \sqrt{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2}$

$= \sin\frac{x}{2} + \cos\frac{x}{2}$, if $\sin\frac{x}{2} + \cos\frac{x}{2} \geq 0$

$= -\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)$, if $\sin\frac{x}{2} + \cos\frac{x}{2} \leq 0$

Now, $\sin\frac{x}{2} + \cos\frac{x}{2} \geq 0$

$\Rightarrow \cos\left(\frac{x}{2} - \frac{\pi}{4}\right) \geq 0$

$\Rightarrow -\frac{\pi}{2} \leq \frac{x}{2} - \frac{\pi}{4} \leq \frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$

Similarly, $\sin\frac{x}{2} + \cos\frac{x}{2} \leq 0$

$\Rightarrow \cos\left(\frac{x}{2} - \frac{\pi}{4}\right) \leq 0$

$\Rightarrow \frac{\pi}{2} \leq \frac{x}{2} - \frac{\pi}{4} \leq \frac{3\pi}{2}$

$\Rightarrow \frac{3\pi}{2} \leq x \leq \frac{7\pi}{2}$

Again, $\sqrt{1-\sin x} = \sqrt{\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)^2}$

$= \sin\frac{x}{2} - \cos\frac{x}{2}$, if $\sin\frac{x}{2} - \cos\frac{x}{2} \geq 0$

$= \cos\frac{x}{2} - \sin\frac{x}{2}$, if $\cos\frac{x}{2} - \sin\frac{x}{2} \geq 0$

Now, $\sin\frac{x}{2} - \cos\frac{x}{2} \geq 0$

$\Rightarrow \cos\left(\frac{x}{2} + \frac{\pi}{4}\right) \leq 0$

$\Rightarrow \frac{\pi}{2} \leq \frac{x}{2} + \frac{\pi}{4} \leq \frac{3\pi}{2}$

$\Rightarrow \frac{\pi}{4} \leq \frac{x}{2} \leq \frac{5\pi}{4}$

$\Rightarrow \frac{\pi}{2} \leq x \leq \frac{5\pi}{2}$

Similarly,

$\cos\frac{x}{2} - \sin\frac{x}{2} \geq 0$

$\Rightarrow \cos\left(\frac{x}{2} + \frac{\pi}{4}\right) \geq 0$

$$\Rightarrow -\frac{\pi}{2} \le \frac{x}{2} + \frac{\pi}{4} \le \frac{\pi}{2}$$

$$\Rightarrow -\frac{3\pi}{4} \le \frac{x}{2} \le \frac{\pi}{4}$$

$$\Rightarrow -\frac{3\pi}{2} \le x \le \frac{\pi}{2}$$

We thus have the following results:

$$\sqrt{1+\sin x} = \sin\frac{x}{2} + \cos\frac{x}{2}, \quad \text{if } -\frac{\pi}{2} \le x \le \frac{3\pi}{2} \quad \text{....(i)}$$

$$= -\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right), \quad \text{if } \frac{3\pi}{2} \le x \le \frac{7\pi}{2} \quad \text{....(ii)}$$

$$\sqrt{1-\sin x} = \sin\frac{x}{2} - \cos\frac{x}{2}, \quad \text{if } \frac{\pi}{2} \le x \le \frac{5\pi}{2} \quad \text{....(iii)}$$

$$= \cos\frac{x}{2} - \sin\frac{x}{2}, \quad \text{if } -\frac{3\pi}{2} \le x \le \frac{\pi}{2} \quad \text{....(iv)}$$

From (i) and (ii),

$$\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} = \tan\frac{x}{2}, \quad \text{if } -\frac{\pi}{2} \le x \le \frac{3\pi}{2}$$

and $\frac{\pi}{2} \le x \le \frac{5\pi}{2}$

$$\Rightarrow \frac{\pi}{2} \le x \le \frac{3\pi}{2}$$

In terms of general values,

$$2n\pi + \frac{\pi}{2} \le x \le 2n\pi + \frac{3\pi}{2} \quad (n = 0, \pm1, \pm2, \ldots\ldots\ldots\ldots)$$

From (ii) and (iv),

$$\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} = \tan\frac{x}{2}, \quad \text{if } \frac{3\pi}{2} \le x \le \frac{7\pi}{2}$$

and $-\frac{3\pi}{2} \le x \le \frac{\pi}{2}$

which is absurd.

From (i) and (iv),

$$\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} = \cot\frac{x}{2}, \quad \text{if } -\frac{\pi}{2} \le x \le \frac{3\pi}{2}$$

and $-\frac{3\pi}{2} \le x \le \frac{\pi}{2}$

$$\Rightarrow -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

In terms of general values,

$$2n\pi - \frac{\pi}{2} \le x \le 2n\pi + \frac{\pi}{2} \quad (n = 0, \pm1, \pm2, \ldots..)$$

From (ii) and (iii),

$$\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} = \cot\frac{x}{2}, \quad \text{if } \frac{3\pi}{2} \le x \le \frac{7\pi}{2}$$

and $\frac{\pi}{2} \le x \le \frac{5\pi}{2}$

$$\Rightarrow \frac{3\pi}{2} \le x \le \frac{5\pi}{2}$$

which is included in the general values of x given above.

We thus have

$$\cot^{-1}\left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right)$$

$$= \cot^{-1}\tan\frac{x}{2}$$

$$= \cot^{-1}\cot\left(\frac{\pi}{2} - \frac{x}{2}\right)$$

$$= \frac{\pi}{2} - \frac{x}{2}, \quad \text{if } 2n\pi + \frac{\pi}{2} \le x \le 2n\pi + \frac{3\pi}{2}$$

and the above expression

$$= \cot^{-1}\cot\frac{x}{2} = \frac{x}{2}, \quad \text{if } 2n\pi - \frac{\pi}{2} \le x \le 2n\pi + \frac{\pi}{2}.$$

We thus see that although both the results shown in the first and the second methods are true, they are valid for different sets of values of x. We can get the same sets of values of x, if we solve the original problem in the third method in the following manner:

$$\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$$

$$=\cot^{-1}\left(\frac{1+\sqrt{\cos^2 x}}{\sin x}\right)$$ (Rationalising the denominator as in the third method)

$$=\cot^{-1}\left(\frac{1+\cos x}{\sin x}\right),\quad \text{if}$$

$$\cos x \geq 0 \Rightarrow 2n\pi-\frac{\pi}{2}\leq x\leq 2n\pi+\frac{\pi}{2}$$

$$=\cot^{-1}\cot\frac{x}{2}=\frac{x}{2},\quad \text{if } 2n\pi-\frac{\pi}{2}\leq x\leq 2n\pi+\frac{\pi}{2},$$

If, however,

$$\cos x\leq 0\Rightarrow 2n\pi+\frac{\pi}{2}\leq x\leq 2n\pi+\frac{3\pi}{2},$$

then $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

$$=\cot^{-1}\left(\frac{1-\cos x}{\sin x}\right)=\cot^{-1}\left(\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}.\cos\frac{x}{2}}\right)$$

$$=\cot^{-1}\tan\frac{x}{2}=\cot^{-1}\cot\left(\frac{\pi}{2}-\frac{x}{2}\right)=\frac{\pi}{2}-\frac{x}{2}$$

Thus we have obtained the same results as before. It is now clear that if we are not careful enough about the validity of some trigonometrical formulas in solving a problem, we are likely to get different answers if we solve the problem by different methods and we shall be unable to explain the cause of these different answers. Of course, there are many problems in which we get a unique solution in whatever methods we solve the problems even if we do not bother about the validity of the formulas etc. For example, to simplify an expression like

$$\sin^{-1}\left(x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right),$$

we can solve the problem in various ways. We can substitute

(i) $x=\sin\theta$ and $\sqrt{x}=\cos\phi$ or

(ii) $x=\cos\theta$ and $\sqrt{x}=\sin\phi$ or

(iii) $x=\sin\theta$ and $\sqrt{x}=\sin\phi$ or

(iv) $x=\cos\theta$ and $\sqrt{x}=\cos\phi$

In each case we shall get the final answer as $\sin^{-1}x-\sin^{-1}\sqrt{x}$. But it is always safe to think about the possible values of x. For example, in the above problem, in order that the given expression is real, it is necessary that $0\leq x\leq 1$. Further, θ, ϕ, as introduced in (i) to (iv) must be such that $0\leq\theta$, $\phi\leq\pi/2$.

In conclusion, it may be stated that the examples, which have been given above are only a few out of many fallacious results which may arise due to inappropriate use of the formulas, results, etc., in trigonometry and it is hoped that these will be of some benefit to the readers.

Reference

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DANGER OF IMPROPER USE OF MATHEMATICAL RESULTS, FORMULAS AND SYMBOLS : LIMIT (CALCULUS)

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A lot of work has been done for the improvement of mathematics teaching in schools. Many textbooks in mathematics are also available in our country, where various mathematical problems have been worked out. But the study of the basic theory of mathematics involving theorems, formulas, symbols, etc., has not been so emphasised. Its consequence is very dangerous. It has been shown by the author [1, 2] with examples how fallacious results may be obtained while solving problems of algebra and trigonometry due to the lack of the basic concept in the theory. With the introduction of Calculus in the schools, it is necessary to point out similar fallacious results in Calculus, which may be obtained due to the improper use of these formulas, theorems, etc. In the present article which deals with the topic 'Limit', it will be shown with examples how the majority of students, while solving a problem, apply theorems and formulas of limit wrongly without considering the applicability of these results. It can be mentioned here that many textbooks in mathematics available in the market now-a-days also contain this type of conceptual errors. One of the purposes of this article is, therefore, to make the teachers and students aware of these errors.

Errors due to the Limit of the Sum of Functions

In finding the limit of a function, we make use of the fundamental theorems of limit without examining whether the theorems are applicable or not.

One of the fundamental theorems is that the limit of the sum of a finite number of functions is equal to the sum of the limits of the functions. In finding the limit of the sum of a number of functions, we generally do not bother about the number of functions and apply the above theorem. As a result, we sometimes get wrong answers. We consider some examples:

Example 1: Find the following limit:

$$\underset{n\to\infty}{Lt}\left(\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\text{.....to n terms}\right).$$

Many students will find the limit by applying the theorem stated above. If we apply this theorem, then we shall get:

Given Limit

$$=\underset{n\to\infty}{Lt}\frac{1}{n}+\underset{n\to\infty}{Lt}\frac{1}{n}+\underset{n\to\infty}{Lt}\frac{1}{n}+\text{...... to n terms}$$

= 0 + 0 + 0 + ... to n terms

= 0.

But unfortunately the limiting value is not 0. It is 1. So, the above method of solution is wrong. The correct solution is as follows:

Given Limit

$= \underset{n\to\infty}{Lt}\left(\frac{1}{n}\times n\right)$, (adding $\frac{1}{n}$, n times)

$= \underset{n\to\infty}{Lt}(1) = 1$.

Now, the question is: "Why is the previous method wrong?" This is because the number of functions is n and as n $\to \infty$ the number of functions also becomes infinite. The sum formula of the limit is applicable only when the number of functions is finite. In the above problem, since the number of functions is not finite, hence the above theorem cannot be applied here in solving the problem.

Example 2: We now consider the following limit:

$$\underset{n\to\infty}{Lt}\left[\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+\ldots..+\frac{n-1}{n^2}\right]$$

Using the sum formula, we get the given limit as

$$\underset{n\to\infty}{Lt}\frac{1}{n^2}+\underset{n\to\infty}{Lt}\frac{2}{n^2}+\underset{n\to\infty}{Lt}\frac{3}{n^2}+\ldots..+\underset{n\to\infty}{Lt}\frac{n-1}{n^2}$$

= 0 + 0 + 0 + ... + 0, (n zeros)

= 0.

Alternatively, if we add the n-1 functions first and then take the limit, we get the given limit as

$$\underset{n\to\infty}{Lt}\frac{(n-1)n}{2n^2}+\underset{n\to\infty}{Lt}\frac{n^2-n}{2n^2}-\underset{n\to\infty}{Lt}\left(\frac{1}{2}-\frac{1}{2n}\right)$$

$$=\frac{1}{2}-0=\frac{1}{2}.$$

The second answer, i.e., $\frac{1}{2}$, is the correct answer and the former answer is wrong, since we get this answer by the wrong use of the sum formula which is applicable in the case of a finite number of terms. The number of terms of the given function is n-1 and as $n\to\infty$, the number of terms becomes infinite. Hence, the sum formula cannot be applied in solving the problem.

We now consider the limit of an infinite series and show that taking the limit term by term of an infinite series leads one to get wrong answer. We explain it with the help of an example.

Example 3: Consider the series:

$x^2+\frac{x^2}{1+x^2}+\frac{x^2}{\left(1+x^2\right)^2}+\ldots\ldots\text{to }\infty$, and suppose that we are interested to find the limit of this series as $x\to 0$.

If we take the limit term by term, i.e., if we apply the sum formula, then we get

$$\underset{x\to\infty}{Lt}\left[x^2+\frac{x^2}{1+x^2}+\frac{x^2}{\left(1+x^2\right)^2}+\ldots\ldots\text{to }\infty\right]$$

$$=\underset{x\to\infty}{Lt}x^2+\underset{x\to\infty}{Lt}\frac{x^2}{1+x^2}+\underset{x\to\infty}{Lt}\frac{x^2}{\left(1+x^2\right)^2}+\ldots\ldots\text{to }\infty$$

= 0 + 0 + 0 + ...to ∞

=0.

But the above answer is wrong due to the fact that we have violated the restriction on the number of terms of an infinite series. The correct solution is, therefore, as follows:

The given series is an infinite G.P. series with common ratio $\frac{1}{1+x^2}$. Hence, the sum of the G.P. series is

$$x^2 \frac{1}{1 - \frac{1}{1+x^2}} = 1 + x^2$$

Hence,

$$\underset{x\to 0}{Lt}\left[x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{\left(1+x^2\right)^2} + \ldots\ldots\text{to } \infty\right]$$

$$= \underset{x\to 0}{Lt}\left(1+x^2\right) = 1.$$

To explain why term by term limit of an infinite series is not allowed, we first state the meaning of the sum of an infinite series.

We consider the following infinite series:

$f_1(x)+f_2(x) + f_3(x) + \ldots\ldots\ldots + f_n(x) + \ldots\ldots\ldots\text{to } \infty$.

Let $Sn(x) = f_1(x) + f_2(x) + \ldots\ldots\ldots\ldots\ldots\ldots + f_n(x)$.

Sn(x), as is well-known, is called the partial sum of the infinite series.

Let $\underset{n\to\infty}{Lt}$ Sn(x) be a finite number, say $n \to \infty$ S. Then S will be called the sum of the series. Hence, the sum of an infinite series is, in fact, the limit of the partial sum Sn(x) as $n \to \infty$ provided this limit is finite.

Now, to explain why we cannot proceed to the limit term by term, we take the following problem involving two limits:

Consider the expression $\frac{x-y}{x+y}$ and suppose that we want to find its limit as x and y both tend to 0. If we let x $\to$ 0 first and then tend y to zero, we shall have

$$\underset{y\to 0}{Lt}\left[\underset{x\to 0}{Lt}\frac{x-y}{x+y}\right] = \underset{y\to 0}{Lt}\left(\frac{-y}{y}\right) = \underset{y\to 0}{Lt}(-1) = -1.$$

If, on the other hand, we let y $\to$ 0 first and then tend x to zero, we shall have

$$\underset{x\to 0}{Lt}\left[\underset{y\to 0}{Lt}\frac{x-y}{x+y}\right] = \underset{x\to 0}{Lt}\frac{x}{x} = \underset{x\to 0}{Lt}(1) = 1.$$

Thus we get two different results.

In the light of the above example, the problem of the limit of the infinite series can be explained.

We have $S(x) = \underset{n\to\infty}{Lt} Sn(x)$.

We now take the limit of the sum function as x $\to$ a, say.

Hence, $\underset{n\to\infty}{Lt}\ S(x) = \underset{x\to a}{Lt}\left[\underset{x\to\infty}{Lt}\ S(x)\right]$, (1)

while the series obtained by taking the limit term by term is

$$\underset{x\to a}{Lt}\ f_1(x) + \underset{x\to a}{Lt}\ f_2(x) + \ldots\ldots\text{to } \infty$$

$$= \underset{n\to\infty}{Lt}\left[\underset{x\to a}{Lt}\ f_1(x) + \underset{x\to a}{Lt}\ f_2(x) + \ldots\ldots + \underset{x\to a}{Lt}\ f_n(x)\right]$$

$$= \underset{n\to\infty}{Lt}\left[\underset{x\to a}{Lt}\ Sn(x)\right] \quad (2)$$

There is no reason to believe that the double limit shown in (1) and (2) will be equal.

There is, however, a special type of infinite series known as power series in which term by term differentiation is possible within the interval of convergence. The discussion of the convergence of this type of series is beyond the scope of the school syllabus. School students should, therefore, be advised not to take the limit of an infinite series term by term in order to avoid possible mistakes.

Errors Due to the Limit of the Product of Functions

Another fundamental theorem of limit is that the limit of the product of two functions is the product of their limits, provided each limit exists. Students generally do not emphasise the restriction involved in the theorem, i.e., existence of each limit.

For example, consider the following limit:

$$\underset{x\to 0}{Lt}\left(x\sin\frac{1}{x}\right)$$

In finding this limit, many students proceed as follows:

$$\underset{x\to 0}{Lt}\left(x\sin\frac{1}{x}\right)$$

$$= \underset{x\to 0}{Lt}\ x \times \underset{x\to 0}{Lt}\ \sin\frac{1}{x} = 0 \times \underset{x\to 0}{Lt}\ \sin\frac{1}{x} = 0.$$

The limiting value is accidentally correct, but the method of solution is wrong due to the fact that $\underset{x\to 0}{Lt}\ x\sin\frac{1}{x}$ does not exist. The theorem on the limit of the product of two functions is applicable only if the limit of each function exists.

The correct method of solution of the above problem can be obtained by giving argument that whatever small value x may have, $\sin\frac{1}{x}$ is always finite (although it is not known definitely) and it lies in the closed interval (-1, 1). Since, $\underset{x\to 0}{Lt}\ x = 0$ hence the limit of x multiplied by a finite number will also be 0 as $x\to 0$.

Students may be curious to know why this restriction is necessary while they get the correct answer even after the violation of the restriction of the theorem. That they may not always get the correct answer by violating this restriction can be shown with an example:

Example 4: Find the value of

$$\underset{x\to a+}{Lt}\ \frac{1}{x-a}\times\left(x^2-a^2\right).$$

If we apply the theorem on the limit of the product of two functions, then we get the given limit as

$$\underset{x\to a+}{Lt}\ \frac{1}{x-a}\times \underset{x\to a+}{Lt}\left(x^2-a^2\right)$$

$= \infty \times 0$ which is indeterminate. Hence, the limit does not exist. But this conclusion is wrong and it will be shown now that this limit exists and hence is finite. The correct answer is as follows:

$$\underset{x\to a^+}{Lt}\ \frac{1}{x-a}\times\left(x^2-a^2\right).$$

$$= \underset{x\to a+}{Lt}\ \frac{(x-a)(x+a)}{x-a}$$

$$= \underset{x\to a+}{Lt}\ (x+a) = 2a$$

which is the correct answer.

The previous method of solution is wrong because $\underset{x\to a+}{Lt}\ \frac{1}{x-a}$ does not exist. We cannot, therefore, use the theorem of the limit of the product of two functions. The above example is one of many to show that students are likely to get incorrect answers due to the violation of the restriction on the above theorems.

Errors Due to Negligence of Symbols

The negligence of certain symbols used in the theory of limit or the lack of knowledge in it leads students to get incorrect answers. Students generally give hardly any importance to the notations like $\underset{x\to a+}{Lt} f(x)$ and $\underset{x\to a-}{Lt} f(x)$.

They sometimes make no distinction between these two concepts although they may be familiar with the symbols representing the right-hand limit and the left-hand limit. As a result, they are likely to get wrong answers. Some examples are given below to explain it.

Example 5: Evaluate , $\underset{x\to 1}{Lt} \sqrt{1-x}$ if possible.

Many students will say that the above limit exists and the limiting value is 0, since 1–x→0 as x→1. But this conclusion is wrong due to the fact that if x >1, then $\sqrt{1-x}$ is imaginary and hence $\underset{x\to 1+}{Lt} \sqrt{1-x}$ does not exist although $\underset{x\to 1-}{Lt} \sqrt{1-x}$ exists and is equal to 0.

The correct answer to the given problem is that the limit does not exist. The mistake done by the students is due to the fact that without examining $\underset{x\to 1+}{Lt} \sqrt{1-x}$ and $\underset{x\to 1-}{Lt} \sqrt{1-x}$.

separately, they simply put x = 1 in $\sqrt{1-x}$ and get a wrong answer.

We consider below another interesting problem where students are likely to do the above type of mistakes due to the negligence of the symbols.

Example 6: It is a well-known result of Coordinate Geometry that the product of the slopes of two perpendicular lines is always -1. Explain whether the result is true for x and y axes which are perpendicular to each other.

A student may analyse the problem like this: The slope of the x-axis is 0 and that of y-axis is ∞. Hence, their product is ∞ × 0 which is indeterminate. Hence, the result is not true for x and y axes. But this conclusion of the student is wrong, since it is a well-known theorem of Coordinate Geometry that the result is true for any pair of perpendicular lines.

A more intelligent student may analyse the problem by the process of limit. He may argue as follows: We denote the x and y axes by Ox and Oy (Fig. 1) and let Ox′ and Oy′ make small angle θ with x and y axes as shown. We shall find the product of the slopes of Ox′ and Oy′ in the limit when θ→0 in which case Ox′ and Oy′ will coincide with x and y axes respectively.

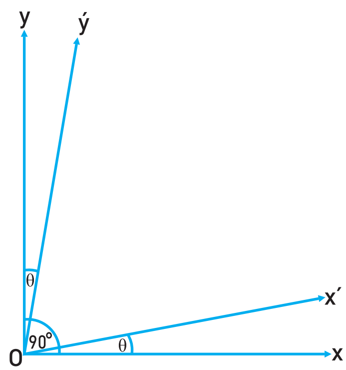

Fig. 1

Now, the slope of Ox′ = tan θ and the slope of Oy′ is tan (90⁰ –θ) = cot θ. The limit of the product of these slopes as θ→0 is $\underset{\theta\to 0}{Lt} \tan\theta\cot\theta = \underset{\theta\to 0}{Lt}(1) = 1,$ which also shows that the result is not true, since the product of the slopes in the limit should be -1.

The above wrong answer is due to the fact that we have not considered whether θ→0 + or, θ→0 –. If, θ→0 + , then Ox′ moves in the clockwise sense and in order that Oy′ tends to coincide with Oy, the rotation of Oy′ should be anticlockwise and hence θ→0 – for Oy′ to coincide with Oy. It is, therefore, evident that while we write θ→0 to find the limit of the product of the slopes of Ox′ and Oy′ we are actually considering $\theta \to 0+$ as well as $\theta \to 0-$ simultaneously, which is absurd.

The figure should, therefore, be drawn in such a manner that Ox′ and Oy′ move in the same direction keeping the angle between them as $\frac{\pi}{2}$ and coincide with Ox and Oy in the limit. This figure is shown below:

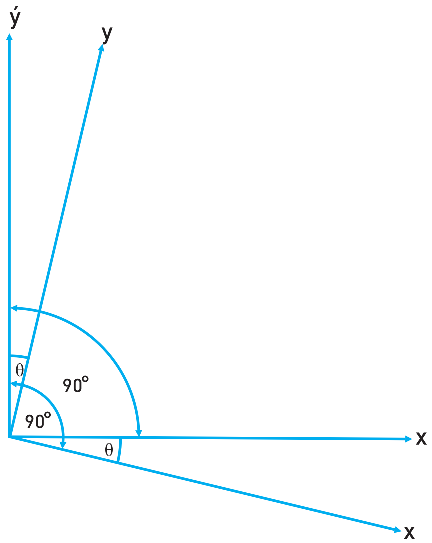

Fig. 2

According to Fig. 2, the slope of Ox′ tan θ and that of Oy′ = tan $(90^0 + \theta) = -\cot\theta$.

Hence, the limit of the product of the slopes of the two lines is $\underset{\theta\to 0+}{Lt} \tan\theta\,(-\cot\theta) = \underset{\theta\to 0+}{Lt}(-1) = -1$. (In order that Ox′ and Oy′ coincide with Ox and Oy respectively, θ should tend to 0 + and not 0–). Hence, the above result is true even for x and y axes also. The above examples will show how dangerous it is to overlook the importance of some symbols in mathematics.

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DANGER OF IMPROPER USE OF MATHEMATICAL RESULTS, FORMULAS AND SYMBOLS : INTEGRALS (CALCULUS)

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The theory of integration is well-known to all mathematicians. Students are also familiar with different results of integration. But this topic is also a subject of serious study like other topics in calculus, which have been explained in a series of articles by the author (1, 2). Students use different standard results of integration in solving a problem without examining their validity. An application of different techniques in finding the integral of a function sometimes gives different results for which students do not find any explanation. Like the previous articles by the same author (1, 2, 3, 4, 5), it will be shown here, with a series of examples, how different results of integration can be obtained due to the lack of knowledge about a theorem and validity of mathematical formulas, etc.

Wrong Use of the Fundamental Theorem of Integral Calculus

Students are familiar with the Fundamental Theorem of Integral Calculus, which is also known as Newton-Leibnitz formula. This theorem can be stated as follows:

$$\int_a^b f(x)dx = F(b) - F(a),$$

Where F(x) is one of the anti-derivatives (or indefinite integrals of f (x) i.e., $F'(x) = f(x)$ in $a \le x \le b$ and f (x) is a continuous function in $a \le x \le b$. It is noticeable that in order that F' (x) exists in $a \le x \le b$, the function F' (x) must also be continuous in $a \le x \le b$. A discontinuous function used as an anti-derivative will lead to the wrong result. Students are familiar with the theorem, but they are not so serious about the nature of the functions f (x) and F (x). The result is, therefore, dangerous, because a student may get the wrong value of an integral, but cannot find any reason for this wrong result. This will be explained with a few examples.

Example 1

Find a mistake in the following evaluation of the integral:

$$\int_0^{\sqrt{3}} \frac{dx}{1+x^2}$$

We know that

$$\frac{d}{dx}\left(\frac{1}{2}\tan^{-1}\frac{2x}{1-x^2}\right) = \frac{d}{dx}\left(\tan^{-1}x\right)$$

$$= \frac{1}{1+x^2}$$

Hence, by the Fundamental Theorem of Integral Calculus, we have

$$\int_0^{\sqrt{3}} \frac{dx}{1+x^2} = \left[\frac{1}{2}\tan^{-1}\frac{2x}{1-x^2}\right]_0^{\sqrt{3}}$$

$$= \frac{1}{2}\left[\tan^{-1}\left(-\sqrt{3}\right) - \tan^{-1}0\right] = -\frac{\pi}{6}$$

Note that the integrand $\frac{1}{1+x^2}$ is positive everywhere in $0 \le x \le \sqrt{3}$, but the area under the curve $y = \frac{1}{1+x^2}$ bounded by the x-axis, the two ordinates x = 0 and x = $\sqrt{3}$ comes out to be negative.

This is physically impossible. What is the mistake then in the evaluation of the integral?

To get an explanation, we should examine whether the Fundamental Theorem of Integral Calculus is applicable for the function.

$\frac{1}{2}\tan^{-1}\frac{2x}{1-x^2}$ in $0 \le x \le \sqrt{3}$

We observe that the function $\frac{1}{2}\tan^{-1}\frac{2x}{1-x^2}$ is discontinuous for x = 1 which lies in $0 \le x \le \sqrt{3}$.

For, $\lim_{x \to 1-} \frac{1}{2}\tan^{-1}\frac{2x}{1-x^2} = \frac{\pi}{4}$ and

$\lim_{x \to 1+} \frac{1}{2}\tan^{-1}\frac{2x}{1-x^2} = -\frac{\pi}{4}$

We have already stated that in order that the Fundamental Theorem of Integral Calculus can be applied to a function in an interval, the function and one of its anti-derivatives should be continuous in that interval. That is why we got an absurd result in the evaluation of an integral.

The correct value of the integral under consideration is equal to $\int_0^{\sqrt{3}} \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^{\sqrt{3}}$

$$= \tan^{-1}\sqrt{3} - \tan^{-1}0 = \frac{\pi}{3},$$

since here $\tan^{-1}x$ is continuous in $0 \le x \le \sqrt{3}$ and the equality F' (x) = f (x) is fulfilled on the whole interval.

Example 2

Evaluate the integral: $\int_0^{\pi} \frac{dx}{1+2\sin^2 x}$

A student may solve the problem as follows:

Let $I = \int \frac{dx}{1+2\sin^2 x}$

Then $I = \int \frac{dx}{\sin^2 x + \cos^2 x + 2\sin^2 x}$

$$= \int \frac{dx}{\cos^2 x + 3\sin^2 x} = \int \frac{\sec^2 x\, dx}{1+3\tan^2 x}$$

Put tan x = z

$\therefore \sec^2 x dx = dz$

$$\therefore I = \int \frac{dz}{1+3z^2} = \frac{1}{3}\int \frac{dz}{z^2 + \left(\frac{1}{\sqrt{3}}\right)^2}$$

$$= \frac{\sqrt{3}}{3}\tan^{-1}\left(\sqrt{3}\,z\right) + C,$$

where C is the constant of integration.

$$\therefore\ I = \frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan x\right) + C$$

Hence, one of the anti-derivatives can be taken as

$$\frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan x\right).$$

Hence, the required integral

$= \frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\sqrt{3}\tan x\right)\right]_0^{\pi} = 0$, a result which is physically impossible, since the integrand $\frac{1}{1+2\sin^2 x}$ is everywhere positive in $0 \le x \le \pi$ and hence the area of the region bounded by the curve y =, x-axis and the two ordinates x = 0 and $x = \pi$ cannot be zero. The student may not find any reason for this mistake in the evaluation of an integral in the above method.

The mistake is due to the fact that the anti-derivative $\frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan x\right)$ is discontinuous at $x = \frac{\pi}{2}$ which lies in the interval $0 \le x \le \pi$. This is because

$$\underset{x \to \frac{\pi}{2}-}{\text{Lt}}\ \frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan x\right) = \frac{1}{\sqrt{3}}.\frac{\pi}{2} = \frac{\pi}{2\sqrt{3}}$$

and

$$\underset{x \to \frac{\pi}{2}-}{\text{Lt}}\ \frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan x\right) = \frac{1}{\sqrt{3}}\left(-\frac{\pi}{2}\right) = \frac{\pi}{2\sqrt{3}}$$

Hence, the Fundamental Theorem of Integral Calculus cannot be applied for this anti-derivative.

The correct value of the integral can be obtained in the following way:

$$\text{Let } I = \int \frac{dx}{1+2\sin^2 x}$$

$$= \int \frac{dx}{\cos^2 x + 3\sin^2 x}$$

$$= \int \frac{\operatorname{cosec}^2 x\,dx}{3+\cot^2 x}$$

Put cot x = z.

$\therefore\ \operatorname{cosec}^2 x\,dx = -dz.$

$$\text{Hence, } I = -\int \frac{dz}{3+z^2}$$

$$= -\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{z}{\sqrt{3}}\right) + C$$

$$= -\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right) + C.$$

Hence, one of the derivatives is $-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right)$.

This function is continuous in $0 \le x \le \pi$. This is because

$$\underset{x \to 0+}{\text{Lt}}\left[-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right)\right] = -\frac{1}{\sqrt{3}}\tan^{-1}\infty = -\frac{\pi}{2\sqrt{3}}.$$

Also, the value of the function at x = 0 is

$$-\frac{1}{\sqrt{3}}\tan^{-1}\left|\left(\frac{\cot x}{\sqrt{3}}\right)\right|_{x=0} = -\frac{\pi}{2\sqrt{3}}$$

Again,

$$\text{Lt}\left[-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right)\right] = -\frac{1}{\sqrt{3}}\tan^{-1}(-\infty) = \frac{\pi}{2\sqrt{3}}.$$

Also, the value of the function at $x = \pi$ is

$$-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right)\Big|_{x=\pi} = \frac{\pi}{2\sqrt{3}}.$$

Further, $-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right)$ is obviously continuous for all x in $0 < x < \pi$. Hence, according to the definition of the continuity of a function in a closed interval, the anti-derivative

$-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right)$ is continuous in $0 \le x \le \pi$.

Thus, the given integral

$$= -\frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right)\right]_0^{\pi}$$

$$= -\frac{1}{\sqrt{3}}\left[\tan^{-1}(-\infty) - \tan^{-1}(\infty)\right]$$

$$= -\frac{1}{\sqrt{3}}\left(-\frac{\pi}{2} - \frac{\pi}{2}\right) = \frac{\pi}{\sqrt{3}}$$ which is the correct value of the integral.

It can be mentioned here that the above integral can also be found with the help of the previous function:

$$F(x) = \frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan x\right).$$

For this purpose, we should divide the interval of integral $(0, \pi)$ into two sub-intervals: $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \pi\right)$ take into consideration the limiting values of the function F(x) as $x \to \frac{\pi}{2}-$ and $x \to \frac{\pi}{2}+$. Then the anti-derivative becomes a continuous function on each of the sub-intervals and Fundamental Theorem of Integral Calculus is applicable.

Thus we have

$$\int_0^{\pi} \frac{dx}{\cos^2 x + 3\sin^2 x}$$

$$= \int_0^{\pi/2} \frac{dx}{\cos^2 x + 3\sin^2 x} + \int_{\pi/2}^{\pi} \frac{dx}{\cos^2 x + 3\sin^2 x}$$

$$= \frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\sqrt{3}\tan x\right)\right]_0^{\pi/2} + \frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\sqrt{3}\tan x\right)\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{\sqrt{3}}\left[\frac{\pi}{2} - 0 + 0 - \left(-\frac{\pi}{2}\right)\right] = \frac{\pi}{\sqrt{3}}.$$

Error due to the Neglect of the Sign of an Expression under a Square Root

Sometimes we ignore the sign of an expression under the square root in an integral throughout the limits of integration and as a result, we come across with a wrong answer. We give an example to explain this.

Example 3

Evaluate: $\int_0^{\pi} \sqrt{\frac{1 + \cos 2x}{2}}\, dx$.

Most of the students will perhaps evaluate this integral in the following manner:

$$\int_0^{\pi/2} \sqrt{\frac{1 + \cos 2x}{2}}\, dx = \int_0^{\pi} \sqrt{\frac{2\cos^2 x}{2}}\, dx$$

$$= \int_0^{\pi} \cos x\, dx = \left[\sin x\right]_0^{\pi} = 0.$$

But the answer is wrong. This is due to the fact that they have not considered two different signs of cos x in $(0, \pi)$. It is well-known that $\cos x \ge 0$ when $0 \le x \le \frac{\pi}{2}$ and $\cos x \le 0$ when $\frac{\pi}{2} \le x \le \pi$. Hence,

$$\sqrt{\cos^2 x} = \cos x \text{ if } 0 \le x \le \frac{\pi}{2}$$

$= -\cos x$ if $\frac{\pi}{2} \le x \le \pi.$

In the above solution, students have taken $\sqrt{\cos^2 x} = \cos x$ throughout the interval $[0, \pi]$ and hence they got the incorrect answer. The correct solution will be as follows:

$$\int_0^{\pi} \sqrt{\frac{1+\cos 2x}{2}}\,dx$$

$$= \int_0^{\pi} \sqrt{\cos^2 x}\,dx$$

$$= \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} (-\cos x)\,dx$$

$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi} = 1 + 1 = 2.$$

With similar arguments, we can show that

$$\int_0^{100\pi} \sqrt{1-\cos^2 x}\,dx = \int_0^{100\pi} \sqrt{2\sin^2 x}\,dx$$

$$= \sqrt{2}\int_0^{100\pi} |\sin x|\,dx = 200\sqrt{2}.$$

Error due to Making No Distinction between Proper and Improper Integrals

Sometimes students do not make any difference between a proper integral and an improper integral and they calculate the improper integral in the same way as they calculate a proper integral. This is sometimes very dangerous because they are likely to get a wrong answer. To explain this, we give an example:

Example 4

Evaluate: $\int_{-1}^{1} \frac{dx}{x^2}$

Without examining whether it is a proper integral or an improper integral, students generally evaluate the integral in the usual way, as discussed below:

$$\int_{-1}^{1} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{-1}^{1} = -1 - 1 = -2.$$

But this answer is wrong, because it will be shown that the integral does not exist. The mistake is due to the fact that the integrand $\frac{1}{x^2}$ is discontinuous at $x = 0$ which lies in $[-1, 1]$. Hence the Fundamental Theorem of Integral Calculus cannot be applied here. The integral is, in fact, an improper integral. The correct solution of the problem is as follows:

$$\int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{0} \frac{dx}{x^2} + \int_{0}^{1} \frac{dx}{x^2}.$$

$$= \underset{\epsilon_1 \to 0+}{Lt} \int_{-1}^{-\epsilon_1} \frac{dx}{x^2} + \underset{\epsilon_2 \to 0+}{Lt} \int_{\epsilon_2}^{1} \frac{dx}{x^2}$$

The function $\frac{1}{x^2}$ is obviously continuous in $-1 \le x \le -\epsilon_1$ as well as in $\epsilon_2 \le x \le 1$ for $\epsilon_1, \epsilon_2 > 0$. Hence, applying the Fundamental Theorem of Integral Calculus, we get

$$\int_{-1}^{-\epsilon_1} \frac{dx}{x^2} = -\left[\frac{1}{x}\right]_{-1}^{-\epsilon_1} = \frac{1}{\epsilon_1} - 1$$

and

$$\int_{\epsilon_2}^{1} \frac{dx}{x^2} = -\left[\frac{1}{x}\right]_{\epsilon_2}^{1} = -1 + \frac{1}{\epsilon_2}$$

Now, since $\underset{\epsilon_1 \to 0+}{Lt} \frac{1}{\epsilon_1}$ and $\underset{\epsilon_2 \to 0+}{Lt} \frac{1}{\epsilon_2}$ do not exist,

Hence the given integral does not exist.

Making the Proper Integral Improper by Some Operation

In order to simplify an integrand, students sometimes make some operations with its numerator and denominator without thinking about the nature of the integrands in the two cases. As a result, they sometimes obtain an absurd result for which they do not get any explanation. We consider an example to explain this type of error.

Example 5

Evaluate: $\int_0^{\pi/2} \frac{dx}{1+\cos x}$

Many students try to evaluate the integral in the following manner:

$$\int_0^{\pi/2} \frac{dx}{1+\cos x} = \int_0^{\pi/2} \frac{1-\cos x}{1-\cos^2 x}dx$$

[Multiplying the numerator and denominator of the integrand by 1-cos x]

$$= \int_0^{\pi/2} \frac{1-\cos x}{\sin^2 x}dx$$

$$= \int_0^{\pi/2} \operatorname{cosec}^2 x \, dx - \int_0^{\pi/2} \cot x \operatorname{cosec} x \, dx$$

$$= -\left[\cot x\right]_0^{\pi/2} + \left[\operatorname{cosec} x\right]_0^{\pi/2}$$

Which is undefined, since cot 0 and cosec 0 are both infinity.

It is noticeable that the original integrand $\frac{1}{1+\cos x}$ is continuous for all x in $0 \le x \le \frac{\pi}{2}$ and hence the integral should exist. The reason for this contradiction is that multiplying numerator and the denominator of the integrand by $1 - \cos x$ which is zero for $x = 0$, we have made the original integrand discontinuous. In other words, we have changed the original proper integral to an improper integral: $\int_0^{\pi/2} \frac{1-\cos x}{1-\cos^2 x}dx$. But these two integrals are completely different. Since a continuous function can never be equated with a discontinuous function by any operation, multiplication of the numerator and the denominator of the original integrand by $1 - \cos x$ in this particular example is not valid. The correct method of solution is, therefore, as follows:

$$\text{Given integral} = \int_0^{\pi/2} \frac{dx}{\cos^2 \frac{x}{2}}$$

$$= \frac{1}{2}\int_0^{\pi/2} \sec^2 \frac{x}{2} dx$$

$$= \left[\tan \frac{x}{2}\right]_0^{\pi/2} = 1.$$

Improper Substitution in a Definite Integral

No student perhaps thinks much about the validity of a substitution in a definite integral. He becomes satisfied whenever a substitution changes an integral to a standard integral. The result is sometimes dangerous. Consider the following example:

Example 6

Evaluate: $\int_{-1}^{1} \frac{dx}{1+|x|}$

A student may substitute $x = t^2$ whence $dx = 2t\,dt$ in order to remove the modulus sign from the integrand. As a result, we observe that when $t = -1$, $x = 1$ and when $t = 1$, $x = 1$ so that the integral becomes $\int_{1}^{1} \frac{2\,tdt}{1+t^2} = 0$, since the upper and the lower limits of the integral are equal. But this result is wrong, since the integrand is greater than zero for all x in $-1 \le x \le 1$ and so we expect a non-zero positive value for the integral. The mistake is due to the fact that the substitution $x = t^2$ is not valid in this case. This is because the substitution $x = t^2$ makes x always positive whereas in the given interval x may be positive or negative in $-1 \le x \le 1$. We can easily evaluate the integral in the following manner:

$$\text{Given integral} = \int_{-1}^{0} \frac{dx}{1+|x|} + \int_{0}^{1} \frac{dx}{1+|x|}$$

$$= -\int_{-1}^{0} \frac{dx}{x-1} + \int_{0}^{1} \frac{dx}{1+x}$$

$$= -\left[\log|x-1|\right]_{-1}^{0} + \left[\log|1+x|\right]_{0}^{1}$$

$$= = \log 2 + \log 2 = 2\log 2.$$

Conclusion

The series of examples given in the series of articles by the author (1, 2) from calculus on limits, derivatives and integrals are only a few of many such examples. Readers themselves may find a lot of such fallacious results in calculus while solving problems. Similar fallacious results in the topics of algebra, coordinate geometry and trigonometry were shown previously by the author (3, 4, 5). Whenever such fallacious results appear in any branch of mathematics, one should recall and think of the possible restrictions involved in a formula, result, theorem, etc., which have been used to solve a problem. The examples cited in the articles will perhaps convince any reader that studies of any theorem, formula, definition, symbol in mathematics need very serious attention. The purpose of the series of such papers under the same heading on different topics of mathematics, which are at present taught in different classes from Classes IX to XII is to make the students and teachers alert about the danger of improper use of mathematical results, formulas, etc. The author will consider his labour to be fruitful even if a few teachers and students of our country are benefitted by the perusal of these articles.

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SRINIVASA RAMANUJAN : THE MATHEMATICIAN

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Introduction

It is an undisputed fact that Srinivasa Ramanujan (1887-1920) was the greatest Indian mathematician of modern times. He was also one of the greatest mathematicians of all times in the world.

For assessing the greatness of Ramanujan, we have to answer the following questions:

- What is the field of investigation of mathematicians? What are the typical problems they discuss?
- How does a mathematician think and work?
- What are the qualities of first-rate mathematician?
- What are the qualities of a great mathematician?

We shall attempt to answer these questions with the help of Ramanujan's work on partitions and try to do this at a level at which any intelligent person with a knowledge of high school mathematics can appreciate the answers.

A mathematician studies *patterns* in number and geometrical forms. The number he studies may be natural numbers, integers, rational numbers, real numbers, complex numbers, ordered pairs of thee numbers, ordered n-triples of these numbers and so on...... The geometrical forms may be curves or surfaces in two or three or n-dimensional spaces and generalisations of these.

A mathematician first works out special cases and then with his experience and intuition, he looks for patterns. He then makes conjectures and verifies them with more special cases. When he is convinced about a pattern, he tries to prove it precisely, logically and rigorously.

Thus a mathematician has to have a strong power of intuition and also a strong power of deductive reasoning.

An applied mathematician observes similar patterns in nature and society and he also needs strong power of intuition and logical deduction.

Some mathematicians have strong intuitive powers and some have strong deductive reasoning powers and some have both.

Both intuition and logical reasoning are natural gifts, but these can be developed by training and concentration.

We can rate every mathematician in the scale of 1 to 100 for intuition and on a similar scale for deductive powers.

Professor Hardy, who in some sense discovered the genius of Ramanujan for the world, tried to rate some mathematician in the first scale for intuition and natural talent for mathematics. He gave himself a rating of 25, his friend and colleague Littlewood a rating of 30, David Hilbert a rating of 80 and Ramanujan a rating of 100.

Hardy, Littlewood and Hilbert may however be rated higher than Ramanujan in the scale for deductive powers. The reason for this can be partly in the long training they had received and partly in the Western traditions of rigorous deductive reasoning. Ramanujan had also begun developing his deductive powers and had he lived a little longer, he might have surpassed them in deductive reasoning powers also. However that was not to be.

There is no doubt that in his genius, his natural talent, his unbelievable intuition for mathematical results and his great powers of concentration, he may be ranked as almost the greatest mathematician of all times.

To illustrate how the mind of a mathematician works and in particular how Ramanujan's mind worked, we consider some examples from theory of partitions.

A Systematic Formula for the Number of Partitions of a Natural Number

A partition of a natural number n is a sequence of non-decreasing positive integers whose sum is n. The total number of partitions of n is denoted by p(n). Thus we have,

one method of finding p(n) is the 'brute force' method. One simply enumerates all partitions. One may not be able to proceed beyond 10 or 20 by this method. One will require a great power of concentration and even then one is likely to miss some partitions. Even if a person has perfect powers of concentration and writes one partition

n	Partitions											P(n)
1	1											1
2	2	11										2
3	3	21	111									3
4	4	31	22	211	1111							5
5	5	41	32	311	221	2111	11111					7
6	6	51	42	411	33	321	3111	222	2211	21111	111111	11

The number p(n) increases fast with n. Thus we have the Table I:

n	10	20	30	40	50	60	70	80
p(n)	42	627	564	37338	204206	966467	4087968	15796474
n	90	100			200	600	1000	5000
p(n)	56634173		190569292		3972999029388	$0(10^{21})$	$0(10^{31})$	$0(10^{75})$

per second, one will take about 126,000 years to write all partitions of 200; and to write all partitions of 5000, the whole age of Universe will not enough!

However mathematicians observe patterns in partitions, develop formulae and can write all partitions of n in a much shorter time. Thus Prof. Hans Raj Gupta prepared tables of partitions upto n = 600 without the use of computers. This was a great achievement in itself.

Ramanujan asked a basic question

"Can we find p(n), without enumerating all the partitions of n?" and he and Hardy gave the first answer,

$$p(n) \simeq \frac{1}{4n\sqrt{3}} e\pi(2n/3)^{\frac{1}{2}} \qquad (1)$$

where π is the ratio of the circumference of a circle to its diameter and has an approximate value of 22/7. The number e is a constant whose approximate value is 2.71828. Also the symbol $\simeq$ stands for 'asymptotically equal to'. This does not give the exact value of p(n), but it gives values which are better and better approximations to p(n) as n increases. In fact, we get the Table II:

It is seen that the difference between the exact and asymptotic values goes on increasing, though the percentage error goes on decreasing. It can be shown that the percentage error becomes smaller and smaller as n becomes larger and larger. In fact, the percentage error can be made as small as we like by making n sufficiently large.

It was a remarkable achievement of Ramanujan to think of this asymptotic formula for p(n). He thought of this remarkable result without any formal training in mathematics beyond the 'Intermediate stage' (equivalent to senior secondary) and without any help from anyone. He learnt mathematics himself; he formulated the problem himself and then in collaboration with Hardy, he obtained the formula for giving the correct order of magnitude of p(n) for large values of n.

Most mathematicians work in the same way in which Ramanujan did. They have to learn the mathematics needed for research themselves, they have to identify significant problems for themselves, they have to make conjectures and then they have to prove them. However they do this with the help of libraries well furnished with books and journals, with the help of research

n	10	20	30	40	50	60
asymptotic value given by (1)	48	492	6080	40081	2175967	1024034
% error	14.26	10.37	8.49	7.35	6.55	5.95
n		70	80	90	100	200
asymptotic value given by (1)	4312796	16607269	5936950	199286739	4.1003717 × 105	
% error	5.50	5.13	4.87	4.57	3.36	

guides, with the help of discussions with peers, with the help of conferences and symposia and so on. What is remarkable about Ramanujan is that he did all this by himself. He had no teacher, no research guide, no library worth the name, no journals, no discussions, no seminars except at a late stage, yet he obtained wonderful results.

Ramanujan-Hardy Formula for the Exact Value of Number of Partitions of a Natural Number

Ramanujan was not satisfied with the asymptotic expression given in (1) (which we may denote by $p_0(n)$ and was convinced that there must be an exact expression for p(n) of the form

$p_0(n) + p_1(n) + p_2(n) + p_3(n) + ...$

where $p_0(n) \gg p_1(n) \gg p_2(n) \gg p_3(n)$ ..... (Here $\gg$) stands for much greater than) so that the terms decrease fast and first few terms may give the correct value for p(n). In fact he and Hardy collaborated together to find this series. For n = 200 their series gives

$p_0(200) = 3972998993185.896$

$p_1(200) =$	+ 36282.978
$p_2(200) =$	- 87.555
$p_3(200) =$	+ 5.147
$p_4(200) =$	+ 1.424
$P_5(200) =$	+ 0.071
	3972999029387.961

Thus six terms are sufficient to give us the exact value. For n = 1000, one may require 15 terms or so, but they showed that the number of terms required will be always of the order $\sqrt{n}$.

The Hardy-Ramanujan theorem for p(n) is one of the most remarkable theorems in mathematics. At this stage we cannot do better than quote Littlewood (Maths Gazette 14, 427-428, 1929).

"The reader does not need to be told that this is a very astonishing theorem and he will readily believe that the method by which it was established involved a new and important principle, which has been found very useful in other fields. The story of the theorem is a romantic one.......One of Ramanujan's Indian conjectures was that the first term of the formula was a very good approximation to p(n). The next step in development, not a great one, was to find the solution as an asymptotic sum of which a fixed number of terms were to be taken, the error being of the order of the next term. But from now to the very end, Ramanujan insisted that much more was true than had been established. "There must be a formula with error O(1)". This was his most important contribution; it was both absolutely essential and most extraordinary. The number of terms was made a function of n; this was a very great step and involved new and deep function-theory methods that Ramanujan obviously could not have discovered by himself. The complete theorem then emerged. But the solution of the final difficulty was impossible without one more contribution from Ramanujan, this time a perfectly characteristic one.......His suggestion of the right form of function to be used was an extraordinary stroke of formal genius without which the complete result can never come into the picture at all. There is indeed a touch of real mystery. Why was Ramanujan so sure about correct functional form needed?

Theoretical insight to be the explanation, had to be of an order hardly to be credited.....There is no escape from the conclusion that the discovery of the correct form was a single stroke of insight. We owe the theorem to a singularly happy collaboration of two men, of quite unlike gifts, in which each contributed the best, most characteristic and most fortunate work that was in him. Ramanujan's genius did have this one opportunity worthy of it".

We have quoted Littlewood extensively for the following reasons:

(i) It shows the extraordinary genius, intuitive powers and great mathematical insight of Ramanujan.

(ii) It shows the interaction between intuition and deductive reasoning in mathematics and shows that each need the other. The image of the mathematics as a purely deductive science is incomplete, if not completely wrong.

(iii) It shows the remarkable phenomenon that to prove a result concerning integers, function theory methods based on the concepts of complex numbers of the form $x + \sqrt{-1}\, y$ were required. Thus, imaginary numbers are essential for proving 'real' results.

(iv) Mathematics is a great intellectual enterprise and the proof of every great result in mathematics involves a great intellectual effort and great intellectual struggle. Most of the proof are published without giving an insight into the struggles which go into obtaining these. Here is an exceptional example.

(v) Hardy-Ramanujan collaboration led not only to the formula for p(n), but also to the development of 'circle method' needed in proving these formulae. The circle method has been fundamental to many other problems of analytical number theory where methods of analysis are used to prove results in number theory.

Ramanujan's Contributions to Congruence Properties of Partition Function

By looking carefully at the tabulated values of p(n) from n = 1 to 200, Ramanujan noted the following patterns:

(i) p(5n + 4) is always divisible by 5 so that p(4), p(9), p(14), p(19),... are divisible by 5.

(ii) p(7n + 5) is always divisible by 7 i.e., p(5), p(12), p(19), p(26),... are divisible by 7.

(iii) p(11n + 6) is always divisible by 11 i.e., p(6), p(17), p(28), p(39),... are divisible by 11.

(iv) p(25n + 24) is always divisible by 25 i.e., p(24), p(49), p(74), p(99),... are divisible by 25.

(v) p(35n + 19) is always divisible by 35 i.e., p(19), p(54), p(89), p(124),... are divisible by 35 and so on .

We are giving in appendix the values of p(n) from n = 1 to 100 and readers will find it interesting to verify the properties given above.

Ramanujan proceeded to prove the first four of these properties and then conjectured the following theorem:

"$p(5^a 7^b 11^c\, n + \lambda)$ is divisible by $5^a 7^b 11^c$ for n = 0, 1, 2, 3...

if $24\lambda - 1$ is divisible by $5^a 7^b 11^c$"

As particular cases of this result, we have

(i) a = 1, b = 0, c = 0 gives p(5n + λ) is divisible by 5 if 24 λ-1 is divisible by 5 i.e., if λ = 4.

(ii) a = 0, b = 1, c = 0 gives p(7n + λ) is divisible by 7 if 24 λ-1 is divisible by 7 i.e., if λ = 5.

(iii) a = 1, b = 0, c = 0 gives p(25n + λ) is divisible by 25 if 24 λ-1 is divisible by 25 i.e., if λ = 24.

(iv) a = 0, b = 2, c = 0 gives p(49n + λ) is divisible by 49 if 24 λ-1 is divisible by 49 i.e., if λ = 47.

The reader will find it interesting to verify it for all the values of n = 1 to 100. In fact Ramanujan found that it was true for all n = 1 to 200. This does not however prove that the conjecture is true. In fact Ramanujan was himself careful to state that "The theorem is supported by all the available evidence, but I have not yet been able to find a general proof".

His attitude was that of a mathematician for whom no amount of empirical evidence is enough. There are results which are correct upto 10^9 or 10^{12} and fail thereafter.

In the present case, the conjecture was proved to be false only after 12 years when in 1930, Chawla noticed from the tables prepared by Hans Raj Gupta that –

p(243) = 1339782599344888 is not divisible by 343.

Now if we put a = 0, b = 3, c = 0 in Ramanujan's conjecture, we have first to find λ so that λ–1 is divisible by 343 and we find 24 X 243 –1 = 5831 = 17 X 343 is divisible by 343 so that λ = 243. If we put n = 0 in Ramanujan's conjecture then p(243) should be divisible by 343 but it is not (you may verify it).

That Ramanujan's conjecture was proved false is no reflection on Ramanujan's genius. In fact such conjectures provide a great incentive for mathematical research.

Ramanujan's conjecture was no exception. Forty-eight years after Ramanujan published his original conjecture, Atkin proved the modified conjecture, viz.,

"p($5^a7^b11^c$ + λ) is divisible by $5^a7^{(b+2/2)}11^c$ if 24λ–1 is divisible by $5^a7^b11^c$".

Thus Ramanujan's conjecture is valid for all positive integral values of a and c, but is valid for b = 1 and 2 only. For higher values of b, p($5^a7^b11^c$ + ë) is divisible not by $5^a7^b11^c$, but by $5^a7[^{(0+2)/2}]11^c$.

For b = 3, λ = 243

For b = 4, λ = 2301

For b = 5, λ = 11905

For b = 6, λ = 112747

Thus the next failure of Ramanujan's conjecture after p(243) will occur for p(2301) and obviously might not have been detected even now.

Thus, here Ramanujan, by his careful observation, intuition and insight had reached very near the truth. The need of reaching the truth led to further developments in mathematics.

This is also the mark of the work of a great mathematician that his work, even when it is not perfect leads to further developments in mathematics and sometimes these developments may be even more fruitful than his successes!

The discovery of the result that 24 λ–1 should be divisible by $5^a7^b11^c$ showed a great and incredible insight. Ramanujan proved many other identities connected with the congruence properties of partitions e.g., one such result he proved was

p(4) + p(9)x + p(14)x² + ...

$$= 5\frac{\left[\left(1-x^{5}\right)\left(1-x^{10}\right)\left(1-x^{15}\right).......\right]^{5}}{\left[\left(1-x\ \right)\left(1-x^{2}\right)\left(1-x^{3}\right).......\right]^{6}}.$$

This result has been considered to be representative of the best of Ramanujan's work by Hardy, who says "If I had to select one formula from all Ramanujan's work, I would agree with Major Macmahen on selecting above".

Rogers – Ramanujan's Identities

We have given an example of Ramanujan's conjecture from partitions which was true upto n = 242 but had to be modified for larger values of n. We give now other examples of Ramanujan's conjectures for which he had done limited verification, but which were found to be true.

Ramanujan gave the identities:

$$1+\sum_{n\geq 1} q^{n^2} \Big/ (1-q)\left(1-q^{2}\right).........\left(1-q^{n}\right)$$

$$= \prod_{n=0}^{\infty}\left(1-q^{5n+1}\right)^{-1}\left(1-q^{5n+1}\right)^{-1}$$

$$1+\sum_{n\geq 1} q^{n+n^2} \Big/ (1-q)\left(1-q^{2}\right).........\left(1-q^{n}\right)$$

$$= \prod_{n=0}^{\infty}\left(1-q^{5n+2}\right)^{-1}\left(1-q^{5n+3}\right)^{-1}$$

Ramanujan had verified that the first fifty terms or so of the power series expansion in q on both sides matched.

Ramanujan sent these identities to Hardy in 1913 and Major MacMahon verified these upto 89th powers of q, but even these do not constitute a proof. Later when Ramanujan went to Cambridge, he found that Rogers had proved these in 1894 issue of Proceedings of London Mathematical Society, but these had been forgotten. Ramanujan's rediscovery of these brought fame to Rogers as well as the two together provided new proof in 1919. Meanwhile I. Schur of Germany proved these independently earlier in 1917 itself. Even more recently the Australian physicist R.J. Baxter rediscovered these while working on a problem of statistical mechanics. Still more recently the American mathematician Andrews and Baxter have given what they have called a motivated proof of these identities.

These identities illustrate the International nature of mathematics and the phenomenon of independent discovery in mathematics. Mathematician from India, England, Germany, Australia and America had independent motivations for proving these identities and the motivation came from consideration of Intuition, Rigour, Motivation, Partitions and Statistical Mechanics!

Concluding Remarks

(i) Ramanujan's contributions to theory of partition and Rogers–Ramanujan identities represent only a small part of his contributions of mathematics. We have chosen these topics because these can be relatively easily explained to the layman and these give us a flavour of Ramanujan's genius. These also illustrate not only Ramanujan's style of creative thinking, but of many lesser mathematicians as well.

(ii) Partition theory has many applications to physics and statistics, but Ramanujan's work is great not because it can have and does have applications, but because it shows the great height to which human genius can reach in meeting great intellectual challenges.

(iii) One of the criteria for measuring the stature of a mathematician is to see his impact on mathematics and mathematicians. Many mathematicians are forgotten even in their life time. Mathematics is developing so fast that 95% of what is created goes into oblivion. However, Ramanujan's work is having terrific impact even one hundred years after his birth. In fact his impact appears to increase every day. Hundreds of mathematicians are working on ideas initiated by him or inspired by him and thousands of papers are being written on his work. With each passing day his stature as a mathematician seems to grow.

(iv) Everybody cannot match his genius, but everybody can be inspired by his great dedication to mathematics, his great insight into mathematics, his hard work, his willingness to learn himself, his constant struggle for originality, his willingness to collaborate with others and his single-minded pursuit of mathematics.

(v) The best way to learn about mathematics and mathematicians is by attempting to solve some problems connected with the work. We give below some easy and some difficult problems:

(a) Find λ's for $0 \le a, b, c \le 5$

(b) Find the order of magnitude of $\sum_{r=1}^{n} r \; p(r)$

(c) Find all values of n below 10^4 for which p (n) will not satisfy Ramanujan's conjecture

(d) Find a formula for $\sum_{r=1}^{n} p(r)$

(e) Using (1) draw the graph of log p(n) against n and show that its slope approaches zero as n approaches infinity.

(f) Verify that Ramanujan's conjecture is true for all values of $n \le 100$.

(g) Verify that Roger-Ramanujan's identities are true up to the 20^{th} power of q.

(h) Show that the statement that n is not divisible by both 2^8 and 5^8 is true upto n=99, 999, 9999 and fails only after this value.

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Appendix

Table of Values of p (n)

n	p(n)	n	p(n)	n	p(n)	n	p(n)
1	1	26	3436	51	239943	76	9289091
2	2	27	3010	52	281589	77	10619863
3	3	28	3718	53	329931	78	12132164
4	5	29	4565	54	386155	79	13848650
5	7	30	5604	55	451275	80	15796476
6	11	31	6842	56	526823	81	18004327
7	15	32	8349	57	614154	82	20506255
8	22	33	10143	58	715230	83	23338469
9	30	34	12310	59	831820	84	26543660
10	42	35	14883	60	966467	85	30167357
11	56	36	17977	61	1121505	86	34262962
12	77	37	21637	62	1300156	87	38887673
13	101	38	26015	63	1505499	88	44108109
14	135	39	31185	64	1741630	89	49995925
15	176	40	37338	65	2012558	90	56634173
16	231	41	44583	66	2323520	91	64112359
17	297	42	53174	67	2679689	92	72533807
18	385	43	63261	68	3087735	93	82010177
19	490	44	75175	69	3554345	94	92669720
20	627	45	89134	70	4087968	95	104651419
21	792	46	105558	71	4697205	96	118114304
22	1002	47	124754	72	5392783	97	133230930
23	1255	48	147273	73	6185689	98	150198136
24	1575	49	173525	74	7089500	99	169229875
25	1958	50	204226	75	8118264	100	190569292

LEARNING THROUGH RIDDLES

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The Throne Carpet

King Vikram Singh was a great lover of art. He encouraged artists, not only those who lived in his kingdom but also those who came from abroad. Once it so happened that a great Kashmiri artist, named Rafi, appeared in his court and showed the king a variety of beautiful carpets he had woven. The king was greatly pleased and ordered Rafi to make a beautiful carpet to cover his throne. Rafi gladly agreed to do so and carefully studied the throne. The first thing which he noted was that the length of each step of the throne was equal but their widths and heights were different, as shown in Fig.1.

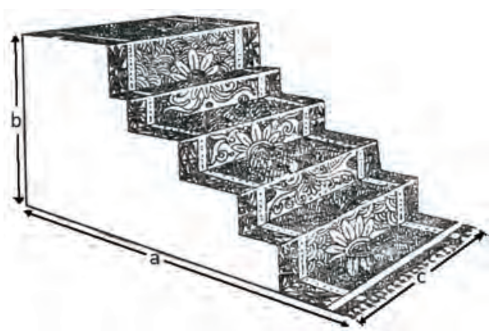

Fig. 1

Rafi, naturally, wanted the various measurements of the throne to know the size of the carpet to be made. Now, according to the royal tradition of the kingdom no one except the king could put his foot on the throne. Also, Rafi could not ask the king himself to measure the length, width and height of each of the steps. He also knew that failure in designing a properly fitting carpet would mean death. A worried Rafi took the king's permission to leave the court; went to the room where he was staying. For the whole night he pondered over the problem and thought of a solution when the cock crowed. He then happily retired to his bed. The following day he came to court and simply measured the lengths a, b and c (see Fig.2). He found a to be 18 ft, b to be 15 ft and c to 6 ft. He then designed a carpet 18+15 = 33 ft long and kept the width of the carpet 6 ft. When he presented the carpet, thus designed, to the king, all the courtiers including the king himself were astonished to see how exactly the beautiful carpet fitted the throne.

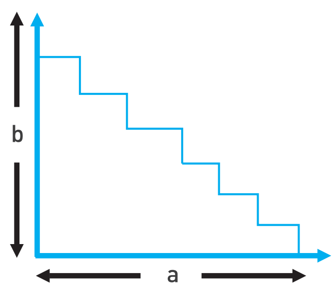

Fig. 2

How did it happen? Which principle helped Rafi in concluding that the length of the needed carpet is simply given by a+b? Let us see. For this purpose look at Fig. 3, where 1 (A S)= b and 1(A G)= a.

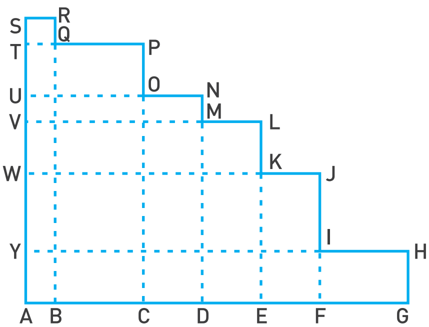

Fig.3

We know that opposite sides of any rectangle are congruent. Hence from Fig. 3 we can say that

$SR \cong AB$

$QP \cong BC$

$ON \cong CD$

$ML \cong DE$

$KJ \cong EF$

$IH \cong FG$

Hence the sum of the lengths of line segments on the left side will be equal to the sum of lengths of the line segments on the right side.

Hence

1 (S R) + 1 (Q P) + 1 (O N) + 1 (M L) + 1 (K L) + 1 (I H)

= 1 (A B) + 1 (B C) + 1 (C D) + 1 (D E) + 1 (E F) + 1 (F G)

= 1(A G) = a..................1.

Similarly,

1 (R Q) = 1 (S T)

1 (P O) = 1 (T U)

1 (N M) = 1 (U V)

1 (L K) = 1 (V W)

1 (J I) = 1 (W Y)

1 (H G) = 1 (Y A)

Hence, adding, we get

1 (R Q) + 1(P O) + 1(N M) +1(L K) +1(J I) +1(H G)=1(S T) +1(T U) +1(U V)+ 1(V W)+ 1(W Y)+ 1 (Y A)

= 1(S A) = b.....2.

Adding 1 and 2, we get

a + b = 1(S R)+1(R Q) + 1(Q P) + 1(P O) + 1(O N)+ 1(N M) + 1(M L) + 1 (L K) + 1(K J)+ 1(J I) + L (I H) + 1 (H G)

= Length of the throne carpet.

This was the calculation which helped Rafi in finding the length of the needed carpet.

Now let us look at this problem from a slightly different angle and see if we stumble across something paradoxical. Let us look at Fig.4.

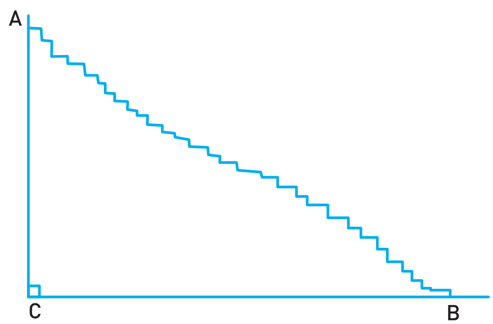

Fig. 4

If we keep on constantly reducing the widths and heights of each step then Fig. 3 gets changed to Fig. 4. AB is thus reduced to a zigzag line. Still the length of this zigzag line is equal to (a + b). Also, we know that as we keep on reducing the size of steps the zigzag line AB resembles more and more a straight line segment AB. In other words, when we reduce the size of each step infinitely, the zigzag line AB becomes a straight line segment. Then our rule tells us that still the length of straight line segment AB should be a + b. But this only means that ΔABC, the sum of the lengths of its two sides is equal to the length of its third side and as we know this is impossible. Then where have we gone wrong to reach this paradox?

Well, the paradox hinges on the fact that we talk of infinity or endless but we always stop the sequence somewhere. We confuse here between infinite number of times and very large number of times. In the above example, one should remember that the length of the zigzag line AB will always be (a + b) no matter how large (but still finite) the number of steps are. But as soon as the number of steps becomes infinite, the rule does not hold and hence the straight line segment AB has a length which is not equal to a + b (but in fact less than a + b).

Catching a Thief

Often, in everyday life, we get indirect information about a variety of things. For example, if a thief steals something and escapes, then police may not learn his name and address directly. Instead, they may get many other small bits of information about the thief. For instance, that he was a tall, fair looking person, or that he was a strong and stout man, etc. Each piece of information thus received brings the police closer and closer to the thief.

This is also true of Mathematics. We often encounter problems where one has to catch the answer hidden in the various pieces of indirect information. For example, instead of telling you directly the number of oranges there are in a store-room, you are told that in the heap 1000 oranges are not good, that 50% of the remaining have been sold and that leaves 3000 oranges. Then one has to find the total number of oranges.

We solve such problems either as a part of our daily work or sometimes we think of them as puzzles for entertainment. Someone not conversant with formal mathematics uses a trial and error method for solving such problems. But a mathematician employs a powerful tool in the solution of such problems. This tool is called an equation. You may wonder what this equation is like. Well, these equations surprisingly resemble our political parties in many respects. Any political party usually pretends to know the answer of peoples' problems. Similarly an equation begins its task by pretending to know the final answer, saying it is X. Now, a political party has its own election symbol such as a lotus or an elephant. So also, an equation has its own symbol which is = (two horizontal line segments). Finally, a political party usually says that it treats rich and poor, literate and illiterate, etc., equally or in other words it is impartial. So also, an equation proceeds on this principle of impartiality, i.e., it always gives identical treatment to its left and right side. i.e., if you divide its right side by 5 then you will have to divide its left side also by 5. If we add 7 to the right side of an equation then we will have to add 7 to its left side too. Using, this principle one can easily solve an equation such as $3x+12=15$. While solving a quadratic equation such as $x^2 + 5x + 6 = 0$, one essentially tries to reduce it to two simple equations $x + 3 = 0$ and $x + 2 = 0$. These equations now can be solved by using the impartiality principle. Even in the solution of simultaneous equations such as $x + y = 5$, $x - 4 = 3$, one tries to reduce them to two simple equations $2x = 8$ and $2y = 2$ which one can solve by using the impartiality principle.

With this much information about the working of equations, let us now turn to a few interesting puzzles recorded in the history of Mathematics. Greek Anthology, Leonardo of Pisa's Liber Abaci and Paciolis Summa give such problems. We will consider here two problems from Anthology and one problem related to the life story of Diophantus who is called the father of Algebra.

(1) Euclid's riddle: A riddle attributed to Euclid and constained in the Anthology is as follows:

Fig.5

A mule and a donkey were walking along laden with corn. The mule says to the donkey, if you gave me one measure I should carry twice as much as you. If I gave you one, we should both carry equal burdens. Tell me their burdens, most learned master of geometry.

Solution: Let the burden carried by the mule be x, and let the burden carried by the donkey be y. Then from the given condition

$(x + 1) = 2(y - 1)$ ...(1)

and $(x - 1) = (y + 1)$ ...(2)

Starting with equation (1)

$x + 1 = 2y - 2$

Take out 1 from both sides.

$\therefore x = 2y - 3$ ...(3)

Adding 1 to both sides of equation (2)

We get $x = y + 2$ ...(4)

From equations (3) and (4) we can say that

$2y - 3 = y + 2$ .... (each being equal to x)

Now adding 3 to both sides we get

$2y = y + 5$

Taking out y from both sides gives

$y = 5$

But from equation (2) $x - 1 = y + 1$

i.e. $x - 1 = 5 + 1$

or $x - 1 = 6$

or $x = 7$

Thus the burden of the mule was 7 units and the burden of the donkey was 5 units.

Tally: $(7 - 1) = 5 + 1$

and $(7 + 1) = 2(5 - 1)$

(2) Diophantus' riddle: In the twilight of the Greek era, Diophantus appeared. Though we do not know the exact period in which he lived, we do know how long he lived. We have this information because one of his admirers described his life in the form of an algebraic riddle. It says.

1. Diophantus' youth lasted 1/6 of his life.

2. He grew a beard after 1/12 more.
3. After 1/7 more of his life Diophantus married.
4. Five year later he had a son.
5. The son lived exactly ½ as long as his father and Diophantus died just four years after his son. How long did Diophantus live?

Solution: Let us suppose that he lived for *x* years.

Hence

(i) His youth was $\frac{x}{6}$

(ii) He grew a beard when $\frac{x}{6}+\frac{x}{12}$ years old.

(iii) He married when he was $\frac{x}{6}+\frac{x}{12}+\frac{x}{7}$ years old.

(iv) He had a son when he was $\frac{x}{6}+\frac{x}{12}+\frac{x}{7}+5$ years old.

(v) His son lived $\frac{x}{2}$ years. Thus when his son died Diophantus was $\frac{x}{6}+\frac{x}{12}+\frac{x}{7}+5+\frac{x}{2}$ years old.

(vi) He died 4 years after his son.

Hence when he died he was

$\frac{x}{6}+\frac{x}{12}+\frac{x}{7}+5+\frac{x}{2}+4$ years old.

(vii) But he lived for x years.

$$\therefore \left(\frac{x}{6}+\frac{x}{12}+\frac{x}{7}+5+\frac{x}{2}+4\right)=x$$

i.e. $5+4=x-\frac{x}{6}-\frac{x}{12}-\frac{x}{7}-\frac{x}{2}$

or $9=\frac{(84\text{-}14\text{-}7\text{-}12\text{-}42)\,x}{84}$

or $9=\frac{9x}{84}$

Multiplying both sides by $\frac{84}{9}$ we get

$\frac{(9)(84)}{9}=x$

or $x = 84$.

Thus Diophantus must have lived for 84 years. Diophantus is called the Father of Algebra because he was the first to abbreviate expression of thoughts with symbols of his own and also because he could solve Indeterminate equations. That is why such equations are often called Diophantine equations.

EXPERIMENTING WITH PYTHAGORAS THEOREM

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It is believed that the theorem was also known to Babylonians a thousand years before Pythagoras in the form of calculating the diagonal of a square. But they associated numbers with lines. Pythagoras was, perhaps, the first to find a proof of the theorem considering the areas of the squares on the sides of a right angle triangle.

Everybody, perhaps, is familiar with the extremely important Euclidean Geometry Theorem: "in any right angle triangle, the square of the hypotenuse is equal to the sum of the squares of other two sides" (Fig.1). This theorem is associated with the Greek Philosopher and Mathematician, Pythagoras who was born in about 580 B.C. in Samos of Greece and later settled in Italy.

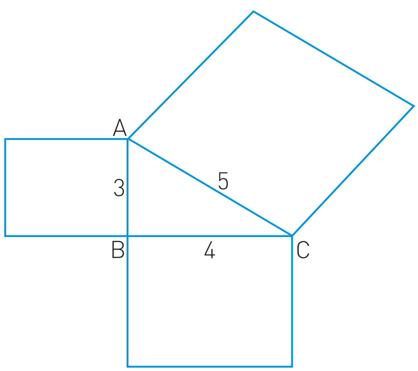

Fig.1

The ancient Indian Baudhayan's Sulba Sutras of 600 B.C. also mentions a similar theorem but in different manner

'dirghacaturasrasyaksnayarajjuh pars vamani tiryanmanica yatprthgbhutekkurutastadubhayam karoti'

which means – 'The diagonal of a rectangle produces by itself both (the areas) produced separately by its two sides' (Fig.2).

This information was also characterized by 'knowledge of plane figures' in another ancient Indian text, Apastamba.

Therefore, the Pythagoras theorem is nothing but the same as described in Sulba Sutras because, the diagonal AC of rectangle ABCD is the hypotenuse of right angle triangle ABC.

It is believed that the theorem was also known to Babylonians a thousand years before Pythagoras in the form of calculating the diagonal of a square. But they associated numbers with lines. Pythagoras was, perhaps, the first to find a proof of the theorem considering the areas of the squares on the sides of a right angle triangle.

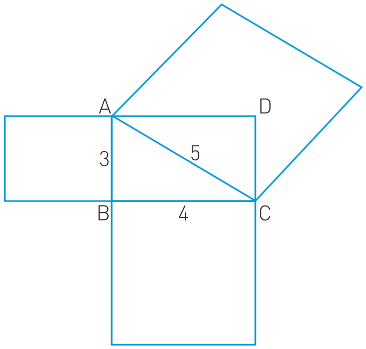

Fig.2

This theorem, still taught as Pythagoras theorem in School Geometry, has now been proved by a number of methods.

Generalisation of Pythagoras Theorem

Pythagoras theorem, taught for squares on the sides so far, can also be proved for any regular polygons of the sides. Therefore, the Pythagoras theorem or Sulba Sutras' knowledge of plane figures could be generalized as follows:

> "The area of any regular polygon on the hypotenuse of a right angle triangle is equal to the sum of the areas of similar regular polygons on other two sides."

Algebraic Proof

In case of equilateral triangle (Fig. 3(a))

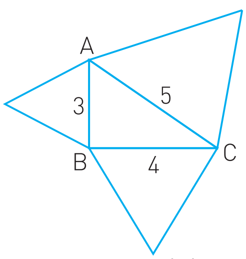

Fig.3 (a)

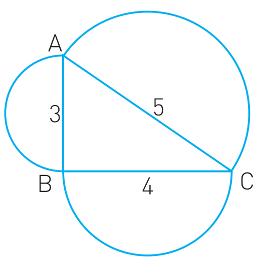

Fig.3 (b)

X = Area of equilateral triangle on

$AC = 25\sqrt{3}/4$

Y = Sum of areas of equilateral triangles on

$AB \text{ and } BC = 16\sqrt{3}/4 + 9\sqrt{3}/4 = 25\sqrt{3}/4$

Therefore, X=Y. Similarly in case of semicircle (Fig.3 (b))

M= Area of semi-circle on

$AC = \frac{1}{2}\pi(5/2)^2 = 25\pi/8$

N= Sum of areas of semi-circles on

$$AB \text{ and } BC = \frac{1}{2}\pi\left(\frac{4}{2}\right)^2 + \frac{1}{2}\pi\left(\frac{3}{2}\right)^2$$

$$= \frac{16}{8}\pi + \frac{9}{8}\pi = \frac{25}{8}\pi$$

Therefore, M = N.

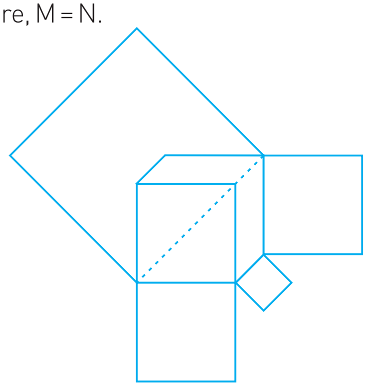

Fig.4

Likewise the theorem can be proved for other regular polygons such as pentagon, hexagon, octagon, etc.

In 3 dimensions also the theorem can be interpreted as 'the area of the square/polygon on the diagonal of the rectangular/cubical lamina is equal to the sum of the areas of squares/polygons on other three sides (Fig.4).

Expansion of the Theorem

The Pythagoras theorem is also true for volumes when the third dimension is same for all the constructed solids on the polygons on the sides of a right angle triangle (Fig5).

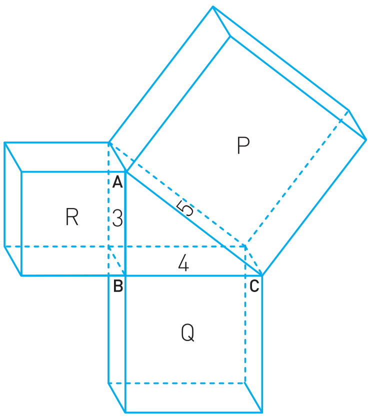

Fig.5

Example

The volume of tray 'P' is equal to the sum of the volume of trays 'Q' and 'R' (depth of all the trays is same).

Proof

Since triangle ABC is a right angle triangle, therefore,

$$AB^2 + BC^2 = AC^2$$

$$m \times AB^2 + m \times BC^2 = m \times AC^2$$

If 'm' be the third dimension (depth) of the trays then the volume of the tray P = volume of the tray Q+ volume of the tray R. However, the theorem cannot hold true for cubes as;

$$AB^3 + BC^3 \neq AC^3.$$

Therefore, the generalized theorem can be stated as follows:

1. The area of any regular polygon on the hypotenuse of a right angle triangle is equal to the sum of the areas of the similar regular polygons on other two sides.
2. The volume of any regular polyhedron constructed on the regular polygon of the hypotenuse of a right angle triangle is equal to the sum of the volumes of similar regular solids, having same third dimension that of the solid on the hypotenuse, constructed on the polygons of the other two sides.

Experimental Proof: A Teaching Aid

The common device one thinks for demonstrating the Pythagoras theorem is by using unit squares to show that the number of unit squares required to fill up the square on hypotenuse side is equal to sum of the unit squares required to fill up the squares on other two sides. However, a more effective aid can be made as follows:

Construction

Make a tray of uniform depth as shown in Fig.6 (a). Fix a piece of the central right triangular size ABC on the tray with openings under the triangle at positions shown by dotted lines for passage of material (remember the trays P, Q and R are square in shape). Fill up mustard seeds or beads in tray 'P' up to the rim. Now cover all trays by square transparent acrylic pieces, cut to the size, with the help of screws. The aid is now ready for demonstration. Hold the aid with tray 'P' on the top. The seeds roll down to tray portions 'Q' and 'R'. One can observe that the seeds in tray portion 'P' will fill up the tray portions 'Q' and 'R' completely. Similarly the aid could be made for other shapes such as equilateral triangle, semi-circle, hexagon and octagon, etc.

This aid can be made more effective using coloured liquids but the material of the tray and sealing has to be properly taken care of.

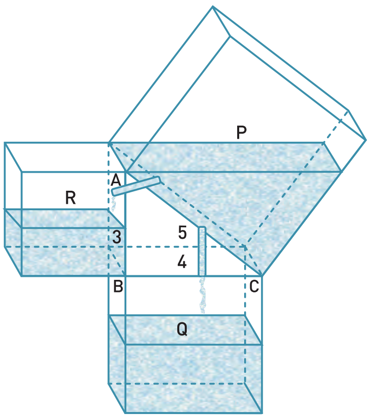

Fig.6(a)

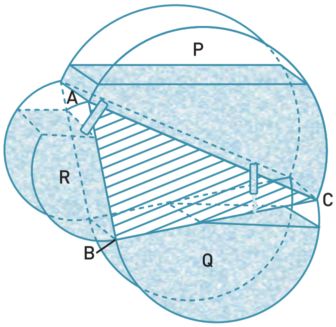

Fig.6(b)

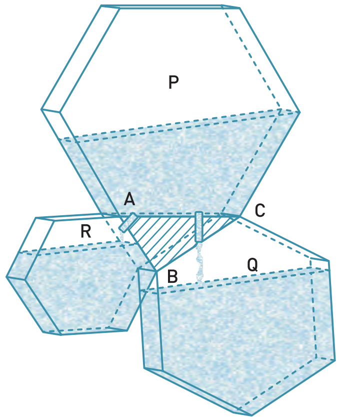

Fig.6(c)

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THE MYSTERIOUS INFINITY

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The concept of endlessness of numbers arises very early when a child in fact learns about the numbers like 1, 2, 3 ... etc., called natural number. A precocious child can argue whether there is any largest natural number. He can build up his argument by assuming that he has found the biggest number N. However, just by adding 1 to N will fetch him another number, N+1, which is still bigger. In this way, he will finally end up with the concept of numerousness of endlessness of numbers.

This endlessness is what the mathematicians have named infinity. It isn't a number like 1, 2 or 3. In fact, it is hard to say what is exactly. It is even harder to imagine what would happen if one tried to manipulate it using the arithmetic operations that work on numbers. For example, what if one multiplies it by 2? Is 1 plus infinity greater than, less than, or the same size as infinity plus 1? What happens if one subtracts 1 from it? These and many other related questions centre round the weird concept of infinity, which has a mysterious realm of its own.

The usual symbol for infinity is ∞. This symbol was first used in a seventeenth century treatise on conic sections. It caught on quickly and was soon used to symbolize infinity or eternity in a variety of contexts.

The appropriateness of the symbol ∞ for infinity lies in the fact that one can travel endlessly around such a curve called leminiscate (Fig.1). Endlessness is, after all, a principal component of one's concept of infinity. Other notions associated with infinity are indefiniteness and inconceivability.

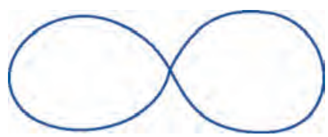


Fig. 1

In the physical world there are various sorts of infinities that could originally exist e.g., infinite time, infinitely large space, infinite-dimensional space, infinitely continuous space etc. Georg Cantor, a German mathematician, has dubbed such infinites physical infinities. In addition to these infinities, Cantor also defines two other types of infinities. These are absolute infinite and mathematical infinities.

Cantor, in fact, allowed for many intermediate levels between the finite and the absolute infinite. These intermediate stages correspond to what Cantor called *transfinite* numbers, that is, numbers that are finite but nonetheless conceivable.

The word 'Absolute' in the context or absolute infinite is used in the sense of 'non-relative, nonsubjective'. An absolute exists by itself and is the highest possible degree of completeness (in religion, this absolute corresponds to God!).

Cantor, introduced the concept of sets and showed that only real numbers can be expressed in terms of infinite sets. A set, as defined by mathematicians, is any collection of distinct or well-defined objects of any sort—people, pencils, numbers. Different objects such as a book, a football, a glass and a table can also constitute a set. The objects constituting a set are called its elements or members. An infinite set is one whose number of elements cannot be counted.

Let us define N as the finite set of all natural numbers:

N = [1, 2, 3, 4,...]

If we remove from N the infinite set of odd natural numbers, then we will be left with the infinite set of even natural numbers:

E = [2, 4, 6, 8, ...]

Similarly, taking away from N the infinite set of even natural numbers leaves us with the set of odd natural numbers:

O = (1, 3, 5, 7...)

So, we see that when an infinite subset is removed from an infinite set the result is an infinite set. Now, consider the infinite set of all natural numbers starting from, say, 11 onwards and remove it from the infinite set of all natural numbers. You will be left with finite set of first ten natural numbers. Thus, removing an infinite subset from an infinite set can also result in a finite subset.

The set of natural numbers and the set of odd (or even) natural numbers are both finite. But, which of the two represents bigger infinity? If we attempt to count the number of odd natural numbers by the usual way, then we find that a one-to-one correspondence exists between the natural numbers and the odd natural numbers. This means that corresponding to every natural number n there exists an odd natural number 2n-1:

1 2 3 4 5,..... n .....

↕↕↕↕↕ ↕

1 3 5 7 9,..... 2n - 1.....

Thus, the number of odd natural numbers is infinite and equal to the number of natural numbers. This is expresses by saying that the cardinality of the set of odd natural numbers is the same as the set of all natural numbers. These two sets, therefore, represent the same infinity.

It can be easily seen that the set of perfect squares has also the same cardinality as the set of natural numbers:

1 2 3 4 5 ..... n, ...

↕↕↕↕ ↕ ↕

1 4 9 16 25,..... n^2,.....

Similarly, the sets of cubes, fourth powers, fifth powers etc., of natural numbers can be seen to have the same cardinality as the set of natural numbers. All these sets, therefore, represent the same infinity.

It can be shown that even the set of all rational numbers has the same cardinality as the set of natural numbers. We can list all the rational numbers by first displaying then in an array. Along the first row we list all those which have numerator 1, along the second row all those with numerator 2, along the third row all those with numerator 3, and so on (Fig.2).

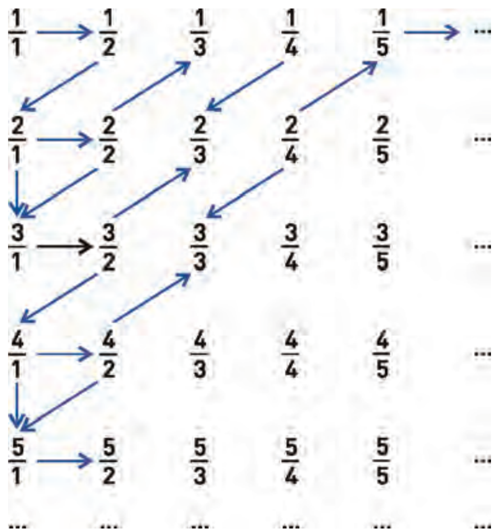
$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$
$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$
$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$	$\frac{4}{5}$ .....
$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{5}{5}$

Fig. 2

Now, if we decide to count row by row then we will be exhausting all the natural numbers by one-to-one correspondence with the entries in the first row. It might seem therefore that because there are an infinity of rows, there must be more rational numbers than natural numbers.

However, we can arrange to count in a different way, beginning with $\frac{1}{1}$ continuing with the rational numbers whose numerator and denominator add upto 3, then with those whose numerator and denominator add upto 4, and so on. This gives us a diagonal method of counting (Fig.3).

It is clear that this arrangement of diagonal counting enables us to list al the natural numbers. The fact that some of them occur more than once in different forms such as $\frac{1}{1},\frac{2}{2},\frac{3}{3},....$ and $\frac{1}{2},\frac{2}{4},\frac{3}{6},....$, does not affect the force of the argument.



In this manner, all the rational numbers can be counted i.e., they can be put into one-to, one correspondence with the natural numbers. There are therefore the same number of rational numbers as there are natural numbers. So, the set of rational numbers has the same cardinality as the set of natural numbers.

We may denote the infinity of natural numbers by N_0 (Cantor used the Hebre letter $\aleph_0$ (aleph-null) to denote this infinity). We say that the set of natural numbers has the same cardinal N_0.

We have seen that the sets of all odd and all even natural numbers have the same cardinality (or infinity) as the set of all natural numbers. Thus, the sets of odd and even natural numbers have N_0 members each. As the sets of odd and even natural numbers together go to make up the set of natural numbers we have: $N_0 + N_0 = N_0$

This defies the dictum applicable to finite collections that 'a part cannot be equal to the whole'. A strange result: isn't it? But, an infinite collection (set) is a collection of objects equal in number to only a part of itself.

Let us now consider a real number i.e., a number expressible in the form:

$\pm n.r_1r_2r_3r_4$

where n is any natural number and r's are digits from 0 to 9. This decimal may be finite and terminating, or may be non-terminating but recurring or may be non-terminating and non-recurring. Cantor expressed real numbers in terms of infinite sets and showed that the real numbers cannot be put in one-to-one correspondence with the natural numbers. Thus, the infinity or cardinality of real numbers is not the same as that of natural numbers.

If we define the infinity of real numbers by N_1 then we can say that N_1 is greater than N_0 provided that we take care in defining precisely what we mean by 'greater than' in the context of infinite numbers. We do this by noting that whereas N_0 numbers can be put in one-to-one correspondence with any part of the N_1 real numbers, there is no way in which N_1 numbers can be put in one-to-one correspondence with a part of N_0 numbers. This applies, of course, to any collection of N_0 and N_1 objects. It is a fundamental truth of what we call 'set theory', the general theory which derives from Cantor's work, and is not confined to collection of numbers.

Numbers such as N_0 and N_1 are called 'transfinite' i.e., they are beyond the finite yet there are infinitely many of them. This was proved by Cantor by showing that given any number, finite or transfinite, it is always possible to construct one that is greater.

Thus, as proved by Cantor, an infinite number of transfinite numbers viz. N_0, N_1, N_2...etc., can exist. We, therefore, have an infinity of infinities. But, can there also exist an infinity between N_0 and N_1, between N_1 and N_2 between N_2 and N_3, and so on. According to Cantor's *continuum hypothesis* there is no infinity in between. Generalising the hypothesis we can say that no infinity can exist between N_k and N_{k+1}.

Cantor's work has now received proper recognition. However, initially, it was put to server criticism. Some of his contemporaries described his set theory as 'a disease', 'repugnant to common sense', and so on. These attacks depressed Cantor and led to a series of nervous breakdowns. This genius dies in 1918 in a mental institution in Halle. The following remarks about infinity made by Cantor deserves special mention:

"The fear of infinity is a form of myopia that destroys the possibility of seeing the actual infinite, even though it in its highest form has created and sustains us, and in its secondary transfinite forms occurs all around us and even inhabits our minds".

THE PRINCIPLE AND THE METHOD OF FINDING OUT THE CUBE ROOT OF ANY NUMBER

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The two methods usually employed to find the cube root of a number are, (i) the method of factorisation, and (ii) the division method. However, it is the method of factorisation that generally finds a place in the school mathematics to obtain the cube root of a number. It may be pointed that this method is convenient to find cube root of the numbers that are a perfect cubes. Here an attempt has been made to explain the method to find the cube root of any number by the method of division. The advantage of this method over the method of factorisation is that it is possible to find the cube root of any number up to the desired decimal places.

Section (A)

The principle of finding out cube root of a number is based on the following known identities and the pattern:

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$= a^3 + (3a^2 + 3ab + b^2)b$ (i)

and $(a + b + c)^3$
$= (a + b)^3 + 3(a + b)^2c + 3(a + b)c^2 + c^3$

$= a^3 + (3a^2 + 3ab + b^2)b + [3(a + b)^2 + 3(a + b)c + c^2]c$ (ii)

also $(a + b + c + d)^3$
$= (a + b + c)^3 + 3(a + b + c)^2d + 3(a + b + c)d^2 + d^3$

$= a^3 + (3a^2 + 3ab + b^2)b + [3(a + b)^2 + 3(a + b)c + c^2]c + [3(a + b + c)^2 + 3(a + b + c)d + d^2]d$ (iii)

on the above pattern, we can write

$(a + b + c + d)^3$

$= a^3 + (3a^2 + 3ab + b^2)b + [3(a + b)^2 + 3(a + b)c + c^2]c + [3(a + b + c)^2 + 3(a + b + c)d + d^2]d + [3(a + b + c + d)^2 + 3(a + b + c + d)e + e^2]e$ (iv)

We see that the identities (i), (ii), (iii) and (iv) have a pattern. Therefore with the help of this pattern we can write cube of any expression.

Section (B)

We can suppose a number in the form of $(a + b)^3$ or $(a + b + c)^3$ or $(a + b + c + d)^3$ etc., and then we find out the value of (a + b), (a + b + c) or (a + b + c + d) as the case be. Thus, the cube root of that number is determined.

In (a + b), a is tens and b is ones.

or if a is ones, then b will be first digit after decimal point.

or if a is first digit after decimal point, then b will be second. In (a + b + c), a is hundreds, b is tens and c is ones.

or if a is tens, then b is ones and c is first digit after decimal point.

or if a is ones, then b and c are first and second digits after decimal points, etc.

Section (C)

We know that $1^3 = 1$, $2^3 = 8$, $3^3 = 27$, $4^3 = 64$ and $5^3 = 125$ .... and $10^3 = 1000$, $20^3 = 8000$, $30^3 = 27000$, $40^3 = 64000$ and $100^3 = 1000000$, $200^3 = 8000000$, $300^3 = 27000000$, $400^3 = 64000000$.

It shows that cube root of a number having 1, 2 or 3 digits will be of one digit.

Cube root of a number, having 4 to 6 digits, will be of two digits.

Cube root of a number, having 7 to 9 digits, will be of three digits.

Section (D)

If a number is multiplied by 1000, then its cube root will be increased by ten times.

If a number is multiplied by 1000000, then its cube root will be increased by hundred times.

If a number is multiplied by 10^9, then its cube root will be increased by thousand times. And so on.

With the help of this property, we can find out cube root of small numbers. This property may also be helpful in finding cube root upto desired number of decimal places.

Section (E)

As it has been already said, Section (B) above that a number can be written in the form of $(a + b)^3$, $(a + b + c)^3$ or $(a + b + c + d)^3$, etc. i.e., in the form of

$a^3 + (3a^2 + 3ab + b^2)b$ (I)

or

$a^3 + (3a^2 + 3ab + b^2)b + [3(a + b)^3 + 3(a + b)c + c^2]c$ (II)

or $a^3 + (3a^2 + 3ab + b^2)b + [3(a + b)^2 + 3(a + b)c + c^2]c$
$+ [3(a + b + c)^2 + 3(a + b + c)d + d^2]d$ (III)

Procedure of Finding Cube Root

(I) We can easily find the value of a as in Section (C). We subtract a^3 from expression (I) above.

The remainder is $(3a^2 + 3ab + b^2)b$

(II) Now putting the value of a in the expression $(3a^2 + 3ab + b^2)$, we try to find out the value of b by division method through approximation.

Now value of a and b are known.

(III) We put the values of a and b in expression $[3(a + b)^2 + 3(a + b)c + c^2]$ and we try to find out value of c by division method through approximation.

This process can be continued for finding values of d and e etc.

Solved Examples

Example 1

To find out the cube root of 3375.

Method

	10+5
10^2	3375
	1000
	2375
	2375
	0

(i) We know $10^3 = 1000$, $1000 < 3375$ and $20^3 = 8000$, $8000 > 3375$

It shows that the cube root of 3375 lies between $3.10^2 + 3.10.5 + 5^2$

10 and 20, therefore $a = 10$ and $a^3 = 1000 = 475$

(ii) Now subtracting 1000 ($10^2 \times 10$) from 3375 remainder is 2375 Cube root of $\therefore 3375 = 10 + 5 = 15$

(iii) To find the value of b we put a = 10 in expression $3a^2 + 3ab + b^2$. We get

$= 3.10^2 + 3.10$ app.

= 330 app.

(iv) If we divide 2375 by 330 we get quotient = 7 app. i.e. b = 7. Putting a = 10 and b = 7 in expression

$(3a^2 + 3ab + b^2)b$, we get $(3.10^2 + 3.10.7 + 7^2)7$

$= (300 + 210 + 49)7 = 459 \times 7$

= 3913

(v) We see that 3913 > 3375

So we put a = 10 and b = 5 in the expression We get $(3.10^2 + 3.10.5 + 25)5$

$= (300 + 150 + 25)5 = 475 \times 5 = 2375$

Therefore b = 5 is the right value.

Cube root of 3375 = 15.

Example 2

Find cube root of 54321

(i) Making groups of three from RHS, we see that 54 is left whose cube root lies between 30 and 40.

So a = 30 and $a^3 = 27000$

Subtracting 27000 from 54321, remainder is 27321.

(ii) Putting a = 30 in exp. $3a^2 + 3ab + b^2$

Value of expression $= 3.30^2 + 3.30$

$= 2700 + 90$

= 2790 app.

Dividing 27321 by 2790,
we see that quotient may be 8 or 9
Putting a = 30 and b = 8 in the expression

$(3a^2 + 3ab + b^2)b$, we get

$(3.30^2 + 3.30.8 + 8^2)8$

$= (2700 + 720 + 64)8$

= 27872   27321

Therefore b = 7 is the correct value

Put b = 7 in the expression

We get $(3.30^2 + 3.30.7 + 7^2)7$

$= 3379 \times 7$

= 23653

Subtracting 23653 from 27321, remainder is 3668

$(3.30^2 + 3.30.7 + 7^2)$

=3379

$$\left[\begin{array}{l} 3(300+70)^2 \\ +3(300+70)(8)+(8^2) \end{array}\right]$$

= 419644

	30+7+.8
30^2	54321
	27000
	27321
	23653
	3668000
	3357152
	310848

(iii) We increase three zeros in remainder 3668 to find out cube root in decimal. Here a, and b will increase 10 times.

i.e., a = 300 and b = 70

Put a = 300 and b = 70 in expression

$3(a + b)^2 + 3(a + b)c + c2$. We get $3(300 + 70)2 + 3(300 + 70)$

$= 3.(370)^2 + 3(370)$

= 410700 + 1110

= 411810

Dividing 3668000 by 411810.
Quotient is 8 or 9

Putting a = 300, b = 70 and c = 8 in the expression $[3(a + b)2 + 3(a + b)c + c^2]c$

We get $[3(300 + 70)^2 + 39300 + 70) (8) + (8^2)](8)$

$= (410700) + (8880 + 64) (8)$

$= 419644 \times 8$

$= 335715 < 3668000$

Therefore c = 8 is correct value.

Cube root of 54321 = 37.8

Note: If we want to find out cube root up to 2 decimal places, we will have to continue the process increasing three zeros in the remainder. Now a, b, c will increase 100 times i.e., a = 3000, b = 700 and c = 80.

Example 3

Find cube root of 15

	2.46
2^2	15
	8
	7000
	5824
	1176000
	1062936
	113064

(i) Cube root of 15 lies between 2 and 3 therefore a = 2 and $a^3 = 8$

ii) Subtracting 8 from 15 remainder is 7. We increase 3 zeroes to find first decimal value. Now a will become 10 times i.e.,

$[3.20^2 + 3.20.4 + 16]$
i.e., a = 20 $= 1456$

(iii) Putting a=20 in the exp. $3a^2+3ab+b^2$, we get $3.(20)^2 + 3.(20) = 1260$ approx.

Now dividing 7000 by 1260, we get quotient =5

Therefore b = 5.

(iv) If we put a = 20 and b = 5 in the expression $[3a^2 + 3ab + b^2]b$, we get

$[3.(20)^2 + 3 \times 20 \times 5 + (5)^2]5$

$= (1200 + 300 + 25)5$

$= 1525 \times 5$

$= 7625$,

which is greater than 7000

Therefore the correct value of b = 4.

Putting a = 20 and b = 4 in expression, we get

$[3.(20)^2 + 3 \times 20 \times 4 + (4)^2]4$

$= 5824$

Subtracting 5824 from 7000, remainder is 1176.

(v) We again increase 3 zeroes in the remainder, now a = 20 and b = 40.

Putting value of a and b in exp.

$3(a + b)^2 + 3(a + b)c + c^2$, we get

$3(200 + 40)^2 + 3(200 + 40)c + c^2$

$= 3 \times 57600 + 720$ approx.

$= 172800 + 720$ approx.

$= 173520$ approx.

Dividing 117600 by 173520, we get value of c = 6.

Putting values of a, b, c in the expression $[3(a + b)^2 + 3(a + b)c + c^2]c$, we get

$[3(200 + 40)^2 + 3(200 + 40).6 + 36]$

$= 3 \times 57600 + 4320 + 36]6$

$= 1771560 \times 6$

$= 1062936$

Subtracting 1062936 from 1176000, remainder is 113064.

Therefore cube root of 15 = 2.46

Example 4

Find cube root of 5 upto two decimal places.

(i) Cube root of 5 lies between 1 + 2

Therefore a = 1 and $a^3 = 1$

Subtracting 1 from 5, remaining is 4.

Increase 3 zeroes to get cube root upto one decimal place. Here a = 10.

Putting a = 10 in expression $(3a^2 + 3ab + b^2)$, we get $3.10^2 + 3.10$ approx.

= 330 approx.

(ii) Divide 4000 by 330, we get b = 9

b = 9 is app. value, putting values of a and b in expression $(3a^2 + 3ab + b^2)$, we get

$[3(10)^2 + 3.10.9 + 9^2)9$

= (300 + 270 + 81)9

= 651 × 9

= 5859 which is far greater than 4000.

Therefore putting b = 7 in the expression, we get

(300 + 210 + 49)7

= 559 × 7

= 3913 < 4000

Therefore value of b = 7

	1.709
1	5
	1
$3.10^2 + 3.10.7 + 49 = 559$	4000
	3913
3(170)2 + 3.(170).0	87000
+0 = 87210	00000
8715981	87000000
	78443829
	8556171

Cube root of 5 = 1.709

(iii) Subtracting 3913 from 4000, remainder is 87. Increasing 3 zeroes in remainder, it becomes 87000. Now value of a = 100 and b = 70. Putting values of a and b in expression $3(a + b)^2 + 3(a + b)c + c^2$, we get $3(100 + 70)^2 + 3(100 + 70)$......approx.

= 3 × 28900 + 510

= 87210 > 87000

It shows that the value of c is less than 1, i.e., value of c = 0.

Putting value of a, b, c in expression, we see that expression becomes zero. Subtracting 0 from 8700 remainder is 87000.

(iv) As we have already seen that 87210 is slightly greater than 87000, therefore now the value d may be 9. Increasing 3 zeroes in remainder it becomes 87000000. Now putting a = 1000, b = 700, c = 00, d = 9 in the expression,

$[3(a + b + c)^2 + 3(a + b + c)d + d^2]d$, we get

$[3(1000 + 700 + 00)^2 + 3(1000 + 700 + 00)(9) + (9)^2]9$

$= [3 \times (1700)^2 + 3 \times 1700 \times 9 + 81]9$

[3 × 2890000 + 5100 × 9 + 81]9

= [8670000 + 45900 + 81]0

= 78443829

Therefore cube root of 5 = 1.709

Note: If the students become familiar with this process, it will be easier to them.

EXPLORING MAHTEMATICS THROUGH ORIGAMI

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Origami literally translates as Ori (folding) gami (paper). It could be used as a valuable method for developing vital skills in mathematics. It is not just a game. All general principles of mathematics are involved in creation of shapes through origami. It can be easily expressed with the help of folding. By using simple square sheets of paper, students can use their imagination and creativity. Difficult and abstract mathematical concepts can be explained in an interesting manner through origami.

The basic tool of origami is a simple sheet of square paper. We will fold a square sheet of paper in any way and begin to see the unlimited possibilities that the square has to offer.

Why is the square sheet of paper used in exploring mathematics in folding?

The square sheet of paper used due to its some peculiar properties:

1. It has all properties of quadrilateral, parallelogram, trapezium and rectangle.
2. Its diagonals are equal in length and bisect each.
3. Its diagonal divides the shape into two isosceles right triangles.
4. The interesting point of diagonals lies on angle bisector of all angles and perpendicular bisector of all sides.
5. All its exterior angles and interior angles are equal.

Objectives

By folding a square sheet of paper, different shapes can be formed. It may represent various aesthetic views of nature. It is done under the scope of ORIGAMI that discloses the hidden ideas of mathematics, when the folded paper is unfolded. Folding and unfolding of the square sheet of paper can be an unending process, which may lead to a new dimension in the teaching of mathematics. Since, this process connects the subject with beauty and real life problems in a practical way, the subject becomes not only creative but also interesting to the teacher and as well as the taught.

I. Folding a Square Sheet of Paper into Half: There are so many ways to half-fold a paper. Every folding creates a number of geometrical shapes. Some of them are described below:

1. **Diaper fold:** A fold made along the diagonal of a square sheet of a paper is called a diaper fold. A diaper fold reveals the shape of two triangles, and the same is shown in Fig.1. Diaper fold is mathematically known as angle bisector fold of a square sheet of paper.

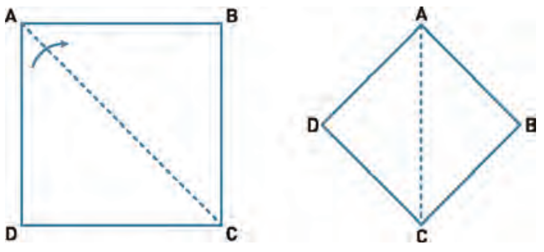

Fig.1

Mathematical Facts Revealed in Diaper Fold

- The triangles formed are congruent.
- The triangles are isosceles right triangles.
- The triangles are equal in area.
- Area of each triangle is half of the area of the square sheet of paper.

2. **Cupboard fold:** A fold made through the perpendicular bisector of any of its side is called a Cupboard Fold. And the folded square sheet of paper turns into a rectangle.

A second cupboard fold, through yet another perpendicular-bisector of the same side, demonstrates the shape of yet another rectangle, but half the size of the first folded rectangles.

And on unfolding, when the paper is seen through its folded-creases, it appears to be divided into four equal parts.

Cupboard fold is mathematically known as perpendicular bisector fold (of any side) of a square sheet of paper. The whole process is shown in Fig. 2.

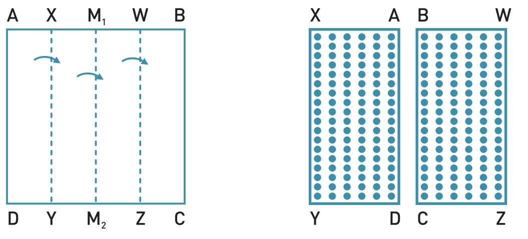

Fig.2

Mathematical Facts Revealed in a Cupboard fold

- The two rectangles formed by the first cupboard-fold are equal in area, with each rectangle as exactly half the square sheet of paper. The ratio of the sides of the rectangles so formed is 1:2.
- The four rectangles formed by the second cupboard-fold, are also equal in area, with each rectangle as exactly one-fourth of the square sheet of paper. Add the ratio of the sides of the rectangles so formed is 1:4.

3. **Blintz fold:** To make this fold, the square sheet of paper is first half-folded to mark the mid-point of all its sides. Once the mid-points are marked, the mid-points of the adjacent sides of the square sheet of paper are joined by further folding, such that the square sheet of paper is turned into a smaller size square. Blintz fold is therefore obtained by joining the mid-points of adjacent sides, by folding the given square sheet of paper. The whole process is shown in Fig.3.

Mathematical Facts Revealed in a Blintz fold

- The area of the small square obtained through this fold is half of the area of the original square sheet of paper.
- The ratio of the sides of the small square so obtained to that of the original paper is $\sqrt{2}:2$ (wherein the value of $\sqrt{2}$ is approximately 1:4).

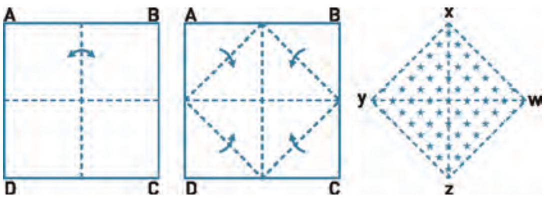

Fig. 3

House - roof fold (Combined with Cupboard-fold)

The square sheet of paper is first half-folded. And then, two adjacent corners are folded to fall on the central crease formed by the half-fold, such that two adjacent isosceles triangles formed on one-half of the square sheet of paper. Now, this half with two adjacent triangles, looks like a house-roof. Next, the other half is a half-folded such that one side of the square falls on a side of both the triangles. The entire process is demonstrated here by Fig.4. The folded paper finally appear with five sides, i.e., as an irregular pentagon (Fig.4.).

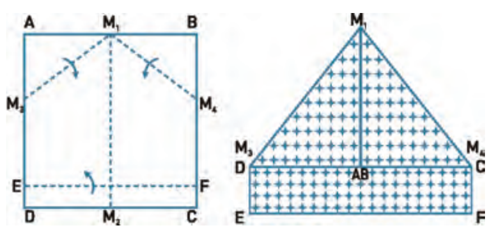

Fig.4

Mathematical Facts Revealed in a House-fold (combined with Cupboard fold)

- The pentagon so formed will have two adjacent sides and two opposite sides as equal.
- Three internal angles of the pentagon will be 90° each and the other two will be 135° each.
- The area of polygon thus obtained will be half the area of the square sheet of paper.
- The folded polygon consists of two isosceles right triangles and one rectangle.
- The equal sides of right triangles are half of the side of square sheet of paper.
- The area of each right triangle will be one-eighth of the area of square sheet of paper.
- The length and breadth of rectangle has ratio of 4:1.
- The area of rectangle is one-fourth of the area of the square sheet of paper.
- The area of rectangle will also be equal to sum of the areas of the two isosceles right triangles.

II. Folding a Square Sheet of Paper into Less or More than Half of its Area: There are also several ways to fold the square sheet of paper to convert it into either less or more than half of its area. Different geometrical shapes thus obtained are explained as follows:

1. House-roof fold: The square sheet of paper is first half-folded. Then, two adjacent corners are folded to fall on the central crease formed by the half-fold, so that the paper now looks like a house-roof. Now, the folded paper has five sides, i.e., like an irregular-pentagon. The entire process is demonstrated through Fig. 5.

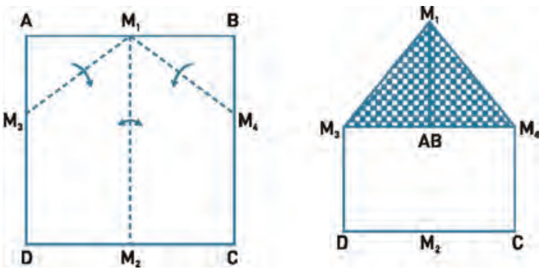

Fig.5

Mathematical Facts Revealed in a House-roof fold

- The pentagon so formed will have two adjacent sides and two opposite sides as equal.
- Three internal angles of the pentagon will be 90° each and the other two will be 135° each.
- The area of polygon thus obtained will be three-fourth of the area of the square sheet of paper.
- The folded polygon consists of two isosceles right triangles and one rectangle.
- The equal sides of right triangles are half of the side of square paper.
- The area of each right triangle will be one-eighth of the area of square sheet of paper.
- The length and breadth of rectangle has a ratio of 2:1.
- The area of rectangle is half the area of the square sheet of paper.

2. Reverse fold: The square sheet of paper is first folded into a diaper fold and then two opposite corners of the diagonal-crease are folded in a manner such that both the opposite sides coincide with diagonal, and the final shape looks like a kite (Fig.6).

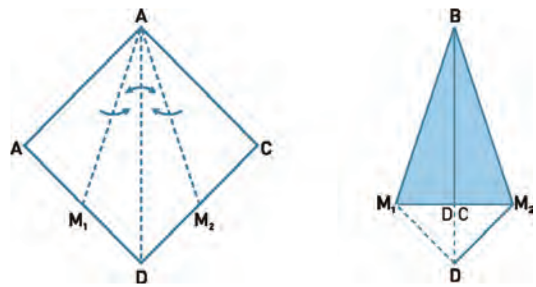

Fig.6

Mathematical Facts Revealed in Reverse fold

- Reverse fold divides the square sheet of paper into four triangles. The two triangles formed in inner region are congruent. Similarly, parts of triangle in outer region of square sheet of paper are also congruent.
- The area of triangle in inner region is $\sqrt{2}$ times the area of triangle in outer region.
- The second and third fold of square sheet of paper divides the side of the paper into $1:\sqrt{2}$ ratio.
- The area of kite is more than half of the area of square paper, i.e., the area of kite is $3/5^{th}$ of the area of square sheet of paper.
- The four interior angles of kite (quadrilateral) are 45°, $112\frac{1}{2}^\circ$.

3. Squash fold: The square sheet of paper is first folded into half and unfolded. This fold made through the perpendicular bisector of any of its side. Again, square sheet of paper is folded in half by another side. Now top right corners are folded forward and backward in such a way that the corner coincides the fold. The final looks like a trapezium (Fig.7).

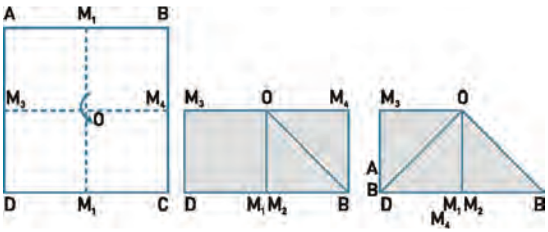

Fig. 7

Mathematical Facts Revealed in Squash fold

- The four interior angles in the trapezium are 45º, 90º, 90º, 135º and parallel sides are in 1:2 ratio.
- The trapezium has right triangle and square shaped figures (or set of three right triangles, among two of them has common hypotenuse).
- In 1st case, we observe that the right triangles are isosceles and side of square is also equal to sides of isosceles right triangle. In second case, we find all three right triangles are isosceles and congruent triangles.
- Thus, the area of trapezium is either equal to the sum of areas of isosceles right triangle and the area of square of half side of square sheet of paper (considering 1st case) or the sum of area of three isosceles right triangles (considering 2nd case). It can be verified by following the area of trapezium.
- The area of above Trapezium - ½ × (X+2X) × X, where X is the half of the side of the square sheet of paper. So we can conclude that the area of trapezium = 3/8 × a, where a is the side of the square.

4. Squash fold (Combined with House-roof fold): If the square sheet of paper is folded by squash fold, again the free end of the paper is folded by House-roof fold. The folded paper finally appears in a parallelogram shape (Fig.8).

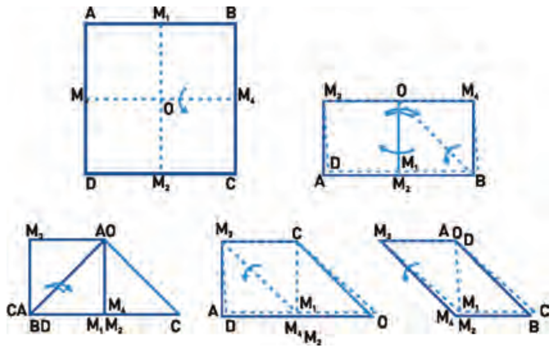

Fig. 8

Mathematical Facts Revealed in a Squash (combined with House-roof fold)

- The parallelogram so formed will have two adjacent sides in 1:$\sqrt{2}$ ratio in lengths.
- The pairs of opposite angles are of 45º and 135º.
- The area of parallelogram is 1/4th the area of the square sheet of paper.
- The ratio of lengths of shorter and longer diagonals of parallelogram is 1:$\sqrt{5}$.

5. Pinch fold: To make this fold, the square sheet of paper is first folded along its each diagonal (i.e., two diaper fold). After unfolding, we get four isosceles triangles in it. Again we fold the square sheet of paper into half along its sides. The folded creases will be the altitudes of the isosceles triangles. Now we fold all adjacent perpendicular bisectors of altitudes and lap the corner on each side of square and crease. This fold is known as Pinch fold. The steps of the fold are explained by the diagram in fig.9.

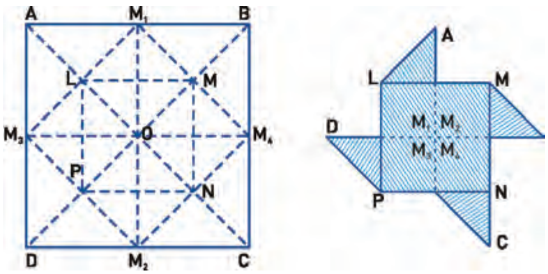

Fig. 9

Mathematical Facts Revealed in a Pinch fold

- The shape obtained by above folding is square with four isosceles right triangles at its four corners.
- The area of square found after folding is 1/4 and area of each right triangle is 1/32 of original square sheet of paper.
- The area of whole shape is 9/32 of the area of original square sheet of paper.
- The sum of areas of four isosceles right triangles is half of the area of its square.
- The area of shape obtained after folding is less than the area of the original square sheet of paper.

6. Petal fold: The square sheet of paper is first folded into half by diaper fold and unfolded to get a centre line. Now right and left corners are folded to the centre line. This is primarily known as reverse fold. Again both opposite corners of the diagonal-crease are folded so that both remaining opposite sides coincide with the diagonal and final shape looks like a rhombus (Fig.10 and Fig.11).

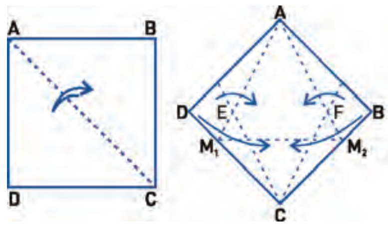

Fig. 10

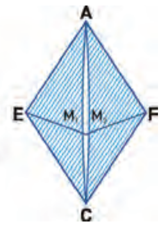

Fig. 11

Mathematical Facts Revealed in Petal fold

- The length of longer diagonal of rhombus found after fold is equal to the length diagonal of the square sheet of paper.
- The length of shorter diagonal of rhombus is less than half of its longer diagonal. The ratio of length of its diagonals is 1:2 (approximately).
- The area of the rhombus is slightly less than half of that of the square sheet of paper i.e., the ratio of areas of rhombus and square sheet of paper is 1:2 (approximately).
- The angles of rhombus are 40°, 135°, 45° and 135°.

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